Perturbations of the Multiple Schramm-Loewner Evolutions Simultaneous Growth Model

Jiaming Chen and Vlad Margarint

September 2021

Abstract

In this work we study multiple SLE_κ, for $\kappa \in (0,4]$, driven by Dyson Brownian motion. This model was introduced in the unit disk by Cardy [18] in connection with the Quantum Calogero-Sutherland model. We study, using a version of Caratheodory convergence, the perturbations on the initial value and parameter $\kappa \in (0,4]$ for the case of $N = 2$ drivers. Our proofs use the analysis of Bessel processes and estimates on Loewner differential equation with multiple drivers. We also study the Hausdorff distance convergence under natural assumptions on the modulus of the derivative of the multiple SLE maps.

1 Introduction

The forward multiple Loewner chain encodes the dynamics of a family of conformal maps defined on simply connected domains of the upper-half plane $\mathbb{H}$. In this work we study a Loewner chain generated by $N \in \mathbb{N}$ continuous driving forces $\{\lambda_1(t), \lambda_2(t), \dots, \lambda_N(t)\}$ from $\mathbb{R}$ to $\mathbb{R}$. We denote these driving functions by $\lambda_j : [0,T] \to \mathbb{R}$, $j = 1, \dots, N$. We have

$$\partial_t g_t(z) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{g_t(z) - \lambda_j(t)}, \tag{1.1}$$

with $g_0(z) = z$. In this paper, we take as driving forces the Dyson Brownian motion. In order to define this object, we consider the Weyl chamber ([2] *Sec.* 4.) defined by

$$\mathcal{M}_N := \{\mathbf{x} \in \mathbb{R}^N;\ x_1 < x_2 < \cdots < x_N\}. \tag{1.2}$$

Let $B_j(t)$, $j = 1, \dots, N$ be one-dimensional standard independent Brownian motions. The Dyson Brownian notions with parameter κ are defined by a system of differential equations in the following

$$d\lambda_j(t) = \frac{1}{\sqrt{2}} dB_j(t) + \frac{2}{\kappa} \sum_{1 \le k \le N, k \ne j} \frac{dt}{\lambda_j(t) - \lambda_k(t)}, \tag{1.3}$$

with $(\lambda_1(0), \dots \lambda_N(0)) \in \mathcal{M}_N$, for all $t \in \mathbb{R}_+$ and $j = 1, \dots, N$. An intuitive picture is that $\{\lambda_1(t), \lambda_2(t), \dots, \lambda_N(t)\}$ describes an ensemble of diffusing particles ([3] *Rmk.* 2.4) in which particles repel each other via a Coulomb force.

It is known that when $\kappa \in (0, 4]$, no two Dyson Brownian particles will collide (*i.e.* touch $\partial \mathcal{M}_N$) almost surely. To be precise, denote by

$$\tau_N := \inf\{0 \le t \le T;\ \exists\ i, j\ s.t.\ |\lambda_i(t) - \lambda_j(t)| = 0\}. \tag{1.4}$$

Then $\tau_N = \infty$ almost surely as in ([16] *Prop.* 3.1). This result also justifies our choice of an arbitrary time interval $[0, T]$. Also, it is known ([1] *Thm.* 1.3) that when $\kappa \in (0, 4]$, the transformations $g_t(z)$ map a simply connected subset $\mathbb{H} \backslash K_t$ conformally onto the upper-half plane $\mathbb{H}$, where K_t consists of the image of N non-intersecting simple curves. Each curve corresponds to a driving force $\lambda_j(t)$, $j \in \mathbb{N}$. We focus on this case and throughout this article we assume $\kappa \in (0, 4]$.

The disk version of the multiple SLE with Dyson Brownian motion driver was introduced by Cardy [18] in connection with the Quantum Calogero-Sutherland Model. The model was recently studied also in [24]. In fact in the last years, there many papers on both the disk and the upper-half plane version of this model (see [18], [19], [20], [21], [22], [23], [25], [26] for a non-exhaustive list). We mainly focus on how the forward Loewner chain $g_t(z)$ behaves when the system is under different perturbations. In the following sections, we propose an estimate of such perturbations in the sense of Carathéodory convergence. Then we study via a version of Caratheodory convergence what happens when either the initial value of the driving forces $\lambda_j(t)'s$ or the parameter κ are perturbed in the $N = 2$ case. The analysis in this paper can be thought as a first-step towards the general N case, and asymptotic $N \to \infty$. In the case $N \to \infty$ there are techniques that involve the study of local statistical properties (such as as the local study of the gaps between particles, and k-point correlations) of the Dyson Brownian motion developed in Random Matrix Theory for the proof of universality of certain random matrix ensembles. We plan to investigate this direction in future work.

The final section is a variant of this analysis in which under a natural assumption on the derivative of the maps at the growth point, we estimate the Hausdorff distance between the perturbation of the hulls K_t.

2 Acknowledgements

We would like to kindly thank Henri Elad-Altman for his valuable comments on analyzing stochastic processes. We also thank A. Swan and L. Schoug for helpful discussions. VM acknowledges the support of the NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai.

3 A version of the Carathéodory Convergence of Loewner Chains

Throughout this paper, we use $\|\cdot\|_{[0,T],\infty}$ for the uniform norm on the interval $[0,T]$, and denote by $\|\cdot\|_{[0,T]\times G}$ the uniform norm on the product space $[0,T]\times G$, where $G \in \mathbb{H}$ is compact. Also, throughout the paper we consider the coupling of the Loewer chains in which both of the chains are driven by Dyson Brownian motion with the same Brownian motions.

In this section, we propose an estimate to the perturbation of forward Loewner chain $g_t(z)$ in the sense of Carathéodory convergence. The central idea is simply convergence on compact sets. This type of convergence is useful. For example in complex analysis, we know ([4] *Thm.* 10.28) that when a sequence of holomorphic functions Carathéodory converges to a limit function, then taking the limit preserves the holomorphicity, hence the limit function is holomorphic. This article follows the convention in ([5] *Sec.* 6.1.1).

In the discussion of forward stochastic Loewner chains $g_t(z) : \mathbb{H}\backslash K_t \to \mathbb{H}$, the hulls K_t evolve stochastically. Using the result in ([7] *Lem.* 3.1) tells us that height $K_t \leq 2\sqrt{t}$ for all $t \in [0,T]$. Hence we could simply subtract the box $\mathbb{R} \times [0, iT]$, which we denote by K_0, from $\mathbb{H}$. We know that $K_t \subset K_0$ for all $t \in [0,T]$ almost surely. Thus, one can consider a version of the Caratheodory convergence in the random setting in which we restrict $g_t(z)$ to the simply connected domain $\mathbb{H}\backslash K_0$.

Definition 3.1. *Let $f_n(t,z) : [0,T] \times \mathbb{H}\backslash K_0 \to \mathbb{H}$ be a sequence of conformal maps, and let $f(t,z) : [0,T] \times \mathbb{H}\backslash K_0 \to \mathbb{H}$ be a conformal map. We say f_n*

converges in the Carathéodory sense to f, or $f_n \xrightarrow{Cara} f$, if for each compact $G \subset \mathbb{H} \backslash K_0$, the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on $[0,T] \times G$.

The estimate on the $g_t(z) : \mathbb{H}\backslash K_0 \to \mathbb{H}$ corresponding to different inputs is based on Definition 2.1. We will give a proposition regarding estimating the difference between two forward Loewner chains. For notational convenience, we will write $g_t(z)$ as $g(t,z)$ from now on. Consider two forward Loewner chains $g_1(t,z)$ and $g_2(t,z)$ defined on $[0,T]\times\mathbb{H}\backslash K_0$, driven respectively by N continuous driving forces $\{V_{k,1}(t),\dots,V_{k,N}(t)\}$ with $k = 1,2$. Then we have the following estimate.

Proposition 3.2. *For an arbitrary compact $G \subset \mathbb{H}\backslash K_0$, there exists a constant $C(T,G) > 0$ such that we have*

$$\|g_1(t,z) - g_2(t,z)\|_{[0,T]\times G,\infty} \le C(T,G) \sum_{j=1}^{N} \|V_{1,j}(t) - V_{2,j}(t)\|_{[0,T],\infty}. \tag{3.1}$$

Proof. It is by *Eqn.* (1.1) that we have the constraint

$$\partial_t g_k(t,z) = \frac{1}{N}\sum_{j=1}^{N} \frac{2}{g_k(t,z) - V_{k,j}(t)}, \tag{3.2}$$

with $k = 1,2$. Choose arbitrarily $z_1, z_2 \in G$. Let $\psi(t) \coloneqq g_1(t,z_1) - g_2(t,z_2)$. And we have

$$\begin{aligned}\frac{d}{dt}\psi(t) &= \partial_t g_1(t,z_1) - \partial_t g_2(t,z_2)\\ &= \frac{1}{N}\sum_{j=1}^{N}\left(\frac{2}{g_1(t,z_1) - V_{1,j}(t)} - \frac{2}{g_2(t,z_2) - V_{2,j}(t)}\right)\\ &= \frac{1}{N}\sum_{j=1}^{N} \xi_j(t)\Big(g_1(t,z_1) - V_{1,j}(t) - g_2(t,z_2) + V_{2,j}(t)\Big),\end{aligned} \tag{3.3}$$

where we define

$$\xi_j(t) \coloneqq \frac{-2}{\big(g_1(t,z_1) - V_{1,j}(t)\big)\cdot\big(g_2(t,z_2) - V_{2,j}(t)\big)}, \tag{3.4}$$

for $j = 1,\dots,N$. Additionally we define $D_j(t) \coloneqq V_{1,j}(t) - V_{2,j}(t)$ for each j. Combined with *Eqn.* (2.3), then we have

$$\frac{d}{dt}\psi(t) = \frac{1}{N}\sum_{j=1}^{N}\xi_j(t)\big(\psi(t) - D_j(t)\big). \tag{3.5}$$

At this moment, we observe that

$$\frac{d}{dt}\left(e^{-\frac{1}{N}\sum\limits_{j=1}^{N}\int_0^t \xi_j(s)ds}\cdot\psi(t)\right) = -\frac{1}{N}\sum_{j=1}^{N}\xi_j(t)D_j(t)\cdot e^{-\frac{1}{N}\sum\limits_{j=1}^{N}\int_0^t \xi_j(s)ds}, \tag{3.6}$$

and consequently

$$\psi(t) = e^{\frac{1}{N}\sum\limits_{j=1}^{N}\int_0^t \xi_j(s)ds}\cdot\psi(0) - \frac{1}{N}\sum_{j=1}^{N}\int_0^t du\cdot\xi_j(u)D_j(u)\cdot e^{\frac{1}{N}\sum\limits_{j=1}^{N}\int_0^u \xi_j(s)ds}. \tag{3.7}$$

On the other hand, we have the following inequality

$$\left|e^{\frac{1}{N}\sum\limits_{j=1}^{N}\int_0^t \xi_j(s)ds}\right| \le e^{\frac{1}{N}\sum\limits_{j=1}^{N}\int_0^t |\xi_j(s)|ds}. \tag{3.8}$$

Then, we know that

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{j=1}^{N}\int_0^t du\cdot\xi_j(u)D_j(u)\cdot e^{\frac{1}{N}\sum\limits_{j=1}^{N}\int_0^t \xi_j(s)ds}\right| \\
&\quad\le \frac{1}{N}\sum_{j=1}^{N}\|D_j(t)\|_{[0,T],\infty}\int_0^t du\cdot|\xi_j(u)|e^{\frac{1}{N}\sum\limits_{j=1}^{N}\int_0^u |\xi_j(s)|ds} \\
&\quad\le \left(\sum_{j=1}^{N}\|D_j(t)\|_{[0,T],\infty}\right)\int_0^t du\cdot\frac{1}{N}\sum_{j=1}^{N}|\xi_j(u)|\cdot e^{\frac{1}{N}\sum\limits_{j=1}^{N}\int_0^u |\xi_j(s)|ds} \\
&\quad= \left(\sum_{j=1}^{N}\|D_j(t)\|_{[0,T],\infty}\right)\cdot\left(e^{\frac{1}{N}\sum\limits_{j=1}^{N}\int_0^t |\xi_j(s)|ds}-1\right).
\end{aligned} \tag{3.9}$$

Moreover, by the Cauchy-Schwartz inequality, we have

$$\frac{1}{N}\sum_{j=1}^{N}\int_0^t |\xi_j(s)|ds \le \frac{1}{N}\sum_{j=1}^{N}\sqrt{I_{1,j}\cdot I_{2,j}}, \tag{3.10}$$

where we define $I_{k,j}$ for $k=1,2$ and $j=1,\ldots,N$ in the following

$$I_{k,j} := \int_0^t \frac{2}{\left|g_k(s,z_k)-V_{k,j}(s)\right|^2}ds = \log\frac{\operatorname{Im} z_k}{\operatorname{Im} g_k(t,z_k)}, \tag{3.11}$$

by ([5] *Sec.* 4.2.2). In fact, we have

$$I_{k,j} \le \log\frac{\operatorname{Im} z_k}{\sqrt{\left((\operatorname{Im} z_k)^2-4t\right)^+}}, \tag{3.12}$$

where $x^+ = \max\{x, 0\}$. Since $t \in [0,T]$ and $z_1, z_2 \in G$ where G is compact in $\mathbb{H}\backslash K_0$, we could choose $\delta(G) \coloneqq \mathrm{dist}(G, \mathbb{R}) > 0$ and define $C(T,G) \coloneqq \frac{\delta(G)}{\sqrt{\delta(G)^2 - 4t}}$. Here we have $I_{k,j} \leq \log C(T,G)$ for all k and j. Hence, we know that

$$|\psi(t)| \leq e^{\frac{1}{N}\sum_{j=1}^{N} \log C(T,G)} \cdot |\psi(0)| + \left(\sum_{j=1}^{N} \|D_j(t)\|_{[0,T],\infty} \right) \cdot \big(C(T,G) - 1\big). \quad (3.13)$$

Therefore, we conclude

$$|g_1(t,z_1) - g_2(t,z_2)| \leq C(T,G) \cdot \big(\sum_{j=1}^{N} \|V_{1,j}(t) - V_{2,j}(t)\|_{[0,T],\infty} + |z_1 - z_2| \big), \quad (3.14)$$

for all $t \in [0,T]$ and $z_1, z_2 \in G$. Now choose $z_1 = z_2 = z$ and take supremum over the left side, we arrive at our final result

$$\|g_1(t,z) - g_2(t,z)\|_{[0,T]\times G,\infty} \leq C(T,G) \cdot \sum_{j=1}^{N} \|V_{1,j}(t) - V_{2,j}(t)\|_{[0,T],\infty}. \quad (3.15)$$

□

Remark 3.3. *With slight changes, the above argument could be adapted to the backward Loewner map. And a similar Carathéodory estimate ([5] Lem.* 6.1*) for the deterministic curves could be proposed.*

Remark 3.4. *In ([17] Sec.* 4.7*) we see another way to formulate Carathéodory convergence with respect to deterministic forward Loewner chains. When the conformal maps are driven by deterministic forces, we take K_T, the deterministic slit of $g_1(t,z)$ and consider uniform convergence on $[0,T] \times G$, where G is compact in $\mathbb{H}\backslash G$. In the stochastic Loewner case, we still prove a pathwise inequality similar to Eqn.* (2.1) *with almost the same proof. The only difference is that the constant factor C now depends additionally on $\omega \in \Omega$. This is a second version of Caratheodory convergence in the random setting that can be obtained with the tools we use in this paper.In the first version one can choose compacts independent of the realizations $\omega \in \Omega$, while in the second the compacts are random and depend on $\omega \in \Omega$.*

4 Perturbations

We are particular interested in the forward Loewner map driven by Dyson Brownian motions. From this section, we restrict our attention to the $N = 2$ case.

We plan to study the general N-curve case in future works. When $N = 2$, we have two driving forces $\{\lambda_1(t), \lambda_2(t)\}$ that are interactign diffusions modelled by Dyson Brownian motion. Their evolution is described in the following equation

$$\begin{aligned} d\lambda_1(t) &= \frac{2}{\kappa} \cdot \frac{dt}{\lambda_1(t) - \lambda_2(t)} + \frac{1}{\sqrt{2}} dB_1(t), \\ d\lambda_2(t) &= \frac{2}{\kappa} \cdot \frac{dt}{\lambda_2(t) - \lambda_1(t)} + \frac{1}{\sqrt{2}} dB_2(t), \end{aligned} \tag{4.1}$$

with $\lambda_1(0) = a_1,\ \lambda_2(0) = a_2,\ a_1 > a_2$, where $B_1(t)$ and $B_2(t)$ are independent one-dimensional Brownian motions. Here we consider only the phase $\kappa \in (0, 4]$. In this case, these two particles $\lambda_1(t)$ and $\lambda_2(t)$ never collide on $\mathbb{R}$. In other words, the stopping time defined in *Eqn.* (1.4) satisfies $\tau_2 = \infty$ almost surely as in ([6] *Prop.* 1.) because $n = 1 + \frac{8}{\kappa} \geq 3$ for $\kappa \geq 4$.

Let $X_t \coloneqq \lambda_1(t) - \lambda_2(t)$. Based on the above observations, we know $X_t > 0$ for all $t \in [0, T]$ almost surely. We further observe that

$$dX_t = \frac{4}{\kappa} \cdot \frac{dt}{X_t} + dW_t, \tag{4.2}$$

with $X_0 = a_1 - a_2$ and $W_t \coloneqq \frac{1}{\sqrt{2}}\big(B_1(t) - B_2(t)\big)$ is a Wiener process. Choose $n = 1 + \frac{8}{\kappa}$, then X_t admits the canonical form of n-dimensional Bessel process with

$$dX_t = \frac{n-1}{2} \cdot \frac{dt}{X_t} + dW_t. \tag{4.3}$$

This section discusses two types of perturbations. The first type of perturbation is varying the initial value of driving forces. The second type is varying the parameter κ. The study in both cases involves the analysis of transient Bessel processes with dimension $n \geq 3$.

4.1 Perturbation of the initial value

The first type of perturbation is to slightly change the initial value of $\lambda_k(0)$ for $k = 1, 2$. With the initial value under perturbation, we get a different set of Dyson Brownian motions. Our goal is to estimate the difference of the forward Loewner chains driven by these varying forces. Perturbations of the initial values appear as technique in Random Matrix Theory in the proof of the Universality of Wigner matrices.

To be precise, choose $0 < \epsilon < \frac{1}{3}(a_1 - a_2)$ and select b_k in the ϵ-ball of a_k for $k = 1, 2$ to be the perturbed initial value of the two Dyson Brownian motions. It is obvious then $b_1 > b_2$ and we arrive at another set of perturbed Dyson

Brownian motions $\{\eta_1(t), \eta_2(t)\}$ with

$$\begin{aligned} d\eta_1(t) &= \frac{2}{\kappa} \cdot \frac{dt}{\eta_1(t) - \eta_2(t)} + \frac{1}{\sqrt{2}} dB_1(t), \\ d\eta_2(t) &= \frac{2}{\kappa} \cdot \frac{dt}{\eta_2(t) - \eta_1(t)} + \frac{1}{\sqrt{2}} dB_2(t), \end{aligned} \tag{4.4}$$

with $\eta_1(0) = b_1$ and $\eta_2(0) = b_2$. Notice that the process $\eta_k(t)$ is still driven by the same Brownian motion $B_k(t)$, because we consider perturbation only on the initial value.

In this two-force case, we denote by $g_\lambda(t, z)$ the original forward Loewner chain generated by forces $\{\lambda_1(t), \lambda_2(t)\}$ and by $g_\eta(t, z)$ the perturbed forward Loewner chain generated by forces $\{\eta_1(t), \eta_2(t)\}$. Hence, we have

$$\begin{aligned} \partial_t g_\lambda(t, z) &= \frac{1}{g_\lambda(t, z) - \lambda_1(t)} + \frac{1}{g_\lambda(t, z) - \lambda_2(t)}, \\ \partial_t g_\eta(t, z) &= \frac{1}{g_\eta(t, z) - \eta_1(t)} + \frac{1}{g_\eta(t, z) - \eta_2(t)}, \end{aligned} \tag{4.5}$$

with $g_{\lambda/\eta}(0, z) = z$ for all $z \in \mathbb{H}$. We continue using $X_t = \lambda_1(t) - \lambda_2(t)$ to denote the gap between two interacting Brownian forces $\lambda_k(t),\ k = 1, 2$. As shown in *Eqn.* (3.2), X_t is a Bessel process with dimension $1 + \frac{8}{k}$ and initial value $X_0 = a_1 - a_2$. Denote by $Y_t := \eta_1(t) - \eta_2(t)$ the gap between $\eta_k(t),\ k = 1, 2$. Then Y_t is a Bessel process with the same dimension $n = 1 + \frac{8}{\kappa}$ and satisfies

$$dY_t = \frac{4}{\kappa} \cdot \frac{dt}{Y_t} + dW_t, \tag{4.6}$$

with $Y_0 = b_1 - b_2$. Observe the Bessel processes X_t and Y_t are driven by the same Wiener process W_t. Hence their difference $X_t - Y_t$ satisfies

$$d(X_t - Y_t) = -\frac{4}{\kappa} \cdot \frac{X_t - Y_t}{X_t Y_t} dt. \tag{4.7}$$

Denote by $a := a_1 - a_2$ and $b := b_1 - b_2$. Integrate both sides on *Eqn.* (3.7) and we see that

$$X_t - Y_t = (a - b) \cdot e^{-\frac{4}{\kappa} \int_0^t \frac{1}{X_s Y_s} ds}. \tag{4.8}$$

Notice that we cannot ascertain $X_t - Y_t$ to be whether deterministic at this moment. In fact, the term $\frac{1}{X_t Y_t}$ might evolve stochastically. Still, for $k = 1, 2$, we could observe that

$$\begin{aligned} d\lambda_k(t) - d\eta_k(t) &= \frac{2}{\kappa}\left(\frac{1}{\lambda_k(t)-\lambda_{3-k}(t)} - \frac{1}{\eta_k(t)-\eta_{3-k}(t)}\right)dt \\ &= (-1)^k\frac{2}{\kappa}\cdot\frac{X_t - Y_t}{\big(\lambda_k(t)-\lambda_{3-k}(t)\big)\cdot\big(\eta_k(t)-\eta_{3-k}(t)\big)}dt. \end{aligned} \tag{4.9}$$

Hence, we have

$$d\lambda_k(t) - d\eta_k(t) = (-1)^k(a-b)\frac{2}{\kappa}\cdot e^{-\frac{4}{\kappa}\int_0^t \frac{1}{X_sY_s}ds}\cdot\frac{1}{X_tY_t}dt, \tag{4.10}$$

for $k = 1, 2$. The above equation admits an integral form

$$\begin{aligned} \lambda_k(t) - \eta_k(t) &= a_k - b_k + (-1)^k(a-b)\frac{2}{\kappa}\int_0^t e^{-\frac{4}{\kappa}\int_0^s \frac{1}{X_uY_u}du}\cdot\frac{1}{X_sY_s}ds \\ &= a_k - b_k + \frac{1}{2}(-1)^{3-k}(a-b)\left(e^{-\frac{4}{\kappa}\int_0^t \frac{1}{X_sY_s}ds} - 1\right). \end{aligned} \tag{4.11}$$

At this point, we have an explicit form to $\lambda_k(t) - \eta_k(t)$. Looking back to Proposition 2.3, we naturally want to have an estimate to $g_\lambda(t,z) - g_\eta(t,z)$ in the Carathéodory sense.

Proposition 4.1. *For all $0 < \epsilon < \frac{a}{3}$ and an arbitrary compact $G \subset \mathbb{H}\backslash K_0$, choose $b_k \in \mathbb{R}$ with $|a_k - b_k| < \epsilon$ for $k = 1, 2$. Let $g_\lambda(z)$ and $g_\eta(z)$ be two multiple Loewner chains induced by Dyson Brownian motions $\{\lambda_1(t), \lambda_2(t)\}$ and $\{\eta_1(t), \eta_2(t)\}$, respectively. Suppose $\lambda_k(0) = a_k$ and $\eta_k(0) = b_k$ for $k = 1, 2$. Then there exists a constant $C(T,G) > 0$ such that almost surely we have*

$$\|g_\lambda(t,z) - g_\eta(t,z)\|_{[0,T]\times G,\infty} < 4C(T,G)\cdot\epsilon. \tag{4.12}$$

Proof. At this moment, we already know $X_t, Y_t > 0$ for all $t \in [0,T]$ almost surely. Inspect *Eqn.* (3.11), we know for $k = 1, 2$ that

$$|\lambda_k(t) - \eta_k(t)| \le |a_k - b_k| + \frac{1}{2}|a-b|\cdot\left(1 - e^{-\frac{4}{\kappa}\int_0^T \frac{1}{X_tY_t}dt}\right) < 2\epsilon. \tag{4.13}$$

By Proposition 2.3, we know that

$$\begin{aligned} \|g_\lambda(t,z) - g_\eta(t,z)\|_{[0,T]\times G,\infty} &\le C(T,G)\cdot\sum_{k=1}^{2}\|\lambda_k(t) - \eta_k(t)\|_{[0,T],\infty} \\ &< 4C(T,G)\cdot\epsilon. \end{aligned} \tag{4.14}$$

And the proposition is verified. ∎

So far we have estimated $g_\lambda(t,z) - g_\eta(t,z)$ in the Carathéodory sense under a perturbation of initial value of driving forces. In practice, when we compute a multiple forward Loewner chain driven by Dyson Brownian motions, we could approximate its initial value and it turns out the approximated Loewner chains converge in a version of the Carathéodory sense. Indeed, we have the following result.

Corollary 4.2. *Suppose $g_t(z) : \mathbb{H}\backslash K_0 \to \mathbb{H}$ is a forward Loewner chain generated by two Dyson Brownian motions $\{\lambda_1(t), \lambda_2(t)\}$ with initial value $\lambda_1(0) > \lambda_2(0)$. Suppose there is a sequence of forward Loewner chains $g_t^n(z)$ generated by Dyson Brownian motions $\{\lambda_1^n(t), \lambda_2^n(t)\}$ with $\lambda_1^n(0) > \lambda_2^n(0)$ and approaching initial value $\lambda_k^n(0) \xrightarrow{n} \lambda_k(0)$. Then we have $g_t^n(z) \xRightarrow{Cara} g_t(z)$ for all $t \in [0,T]$ almost surely.*

4.2 Perturbation of the parameter $\kappa \in (0,4]$

The second type of perturbation is with respect to the parameter κ.This type of perturbation is a natural problem and was considered extensively in the one-curve case where many results have been proved in the recent years in various topologies. Remember that we have always chosen $\kappa \in (0,4]$ so that there is no phase transition ([8] *Sec.* 3.) corresponding to the $(1+\frac{8}{\kappa})$-dimensional Bessel process. When there is perturbation, κ is varied and we have a new parameter $\kappa^* \in (0,4]$ such that $\kappa^* \neq \kappa$. The difference in parameter results in different Dyson Brownian motions, and therefore different forward Loewner chains.

To simplify the model, we assume $\kappa^* > \kappa$ without loss of generality. Denote by $\{\lambda_1(t), \lambda_2(t)\}$ the original Dyson Brownian motions. Their dynamics is described in *Eqn.* (3.1) with initial value $\lambda_1(0) = a_1,\ \lambda_2(0) = a_2,\ a_1 > a_2$. Denote by $\{\lambda_1^*(t), \lambda_2^*(t)\}$ the perturbed Dyson Brownian motions. They respect the following equations

$$\begin{aligned} d\lambda_1^*(t) &= \frac{2}{\kappa^*} \cdot \frac{dt}{\lambda_1^*(t) - \lambda_2^*(t)} + \frac{1}{\sqrt{2}} dB_1(t), \\ d\lambda_2^*(t) &= \frac{2}{\kappa} \cdot \frac{dt}{\lambda_2^*(t) - \lambda_1^*(t)} + \frac{1}{\sqrt{2}} dB_2(t), \end{aligned} \tag{4.15}$$

with initial value $\lambda_k^*(0) = \lambda_k(0) = a_k,\ k = 1,2$. We continue to denote by $g(t,z) : [0,T] \times \mathbb{H}\backslash K_0 \to \mathbb{H}$ the original Loewner chain generated by forces $\{\lambda_1(t), \lambda_2(t)\}$. And we denote by $g^*(t,z) : [0,T] \times \mathbb{H}\backslash K_0 \to \mathbb{H}$ the perturbed

Loewner chain generated by $\{\lambda_1^*(t), \lambda_2^*(t)\}$. The evolution respects

$$\partial_t g^*(t,z) = \frac{1}{g^*(t,z) - \lambda_1^*(t)} + \frac{1}{g^*(t,z) - \lambda_2^*(t)}, \tag{4.16}$$

with $g^*(0,z) = z$ for all $z \in \mathbb{H}\backslash K_0$. Denote by X_t the gap between $\lambda_1(t)$ and $\lambda_2(t)$. Then X_t is a $(1+\frac{8}{\kappa})$-dimensional Bessel process with initial value $X_0 = a$. Its evolution is described in *Eqn.* (3.2). At the same time, let $X_t^* := \lambda_1^*(t) - \lambda_2^*(t)$ the gap of the two perturbed driving forces. The the gap respects the following equations

$$dX_t^* = \frac{4}{\kappa^*} \cdot \frac{dt}{X_t^*} + dW_t, \tag{4.17}$$

with $X_0^* = X_0 = a$ and where W_t is the Wiener process defined in *Eqn.* (3.2). Notice here X_t^* is a $(1+\frac{8}{\kappa^*})$-dimensional Bessel process.

Our main goal is to give an probabilistic estimate of $g(t,z) - g^*(t,z)$ in the Carathéodory sense. Following Proposition 2.3, we need first estimate the sup-norm of $\lambda_k(t) - \lambda_k^*(t)$ for $k = 1,2$. Indeed, define $\nu := \frac{4}{\kappa} - \frac{1}{2}$, $\nu^* := \frac{4}{\kappa^*} - \frac{1}{2}$. Before proving Proposition 2.3, we have the following lemma. Elements of this lemma were kindly provided by H. Elad-Altman in a private communication.

Lemma 4.3. *Given a* $(1+\frac{8}{\kappa^*})$*-dimensional Bessel process* X_t^* *and a* $(1+\frac{8}{\kappa})$*-dimensional Bessel process* X_t *with* $4 \geq \kappa^* > \kappa > 0$ *and the same initial value* $X_0^* = X_0 = a > 0$*, we have almost surely that*

$$\sup_{0\leq s\leq t} (X_s^* - X_s)^2 \leq \frac{4t}{\kappa^2}(\kappa^* - \kappa). \tag{4.18}$$

Proof. Observe *Eqn.* (3.2) and *Eqn.* (3.17), we see

$$X_t^* - X_t = \frac{4}{\kappa^*}\int_0^t \frac{ds}{X_t^*} - \frac{4}{\kappa}\int_0^t \frac{ds}{X_t}. \tag{4.19}$$

Using Itô's lemma, we have

$$\begin{aligned} d(X_t^* - X_t)^2 &= 2(X_t^* - X_t) \cdot \left(\frac{4}{\kappa^* X_t^*} - \frac{4}{\kappa X_t}\right)dt \\ &= \frac{4}{\kappa^*\kappa}(\kappa - \kappa^*) \cdot \frac{X_t^* - X_t}{X_t^*}dt + \frac{8}{\kappa}(X_t^* - X_t) \cdot (\frac{1}{X_t^*} - \frac{1}{X_t})dt. \end{aligned} \tag{4.20}$$

At the same time, we have that $(X_t^* - X_t) \cdot (\frac{1}{X_t^*} - \frac{1}{X_t}) \leq 0$ for all $t \in [0,T]$

almost surely. Integrating both sides and we obtain

$$(X_t^* - X_t)^2 \le (\kappa - \kappa^*)\frac{4}{\kappa^*\kappa} \cdot \int_0^t \frac{(X_s^* - X_s)_+}{X_s^*} ds. \tag{4.21}$$

On the other hand, it is obvious that $(X_s^* - X_s)_+ \le X_s^*$. By considering $\kappa^* \le \kappa$, we have the conclusion

$$\sup_{0 \le s \le t} (X_s^* - X_s)^2 \le \frac{4t}{\kappa^2}(\kappa - \kappa^*). \tag{4.22}$$

□

We denote by $S_t = \sup_{0 \le s \le t} W_t$ the supremum Brownian motion. We are ready to state main result.

Proposition 4.4. *Let $g(t,z)$ and $g^*(t,z)$ be two multiple Loewner chains for the parameters $\kappa, \kappa^* \in (0,4]$, respectively. Choose an arbitrary compact $G \subset \mathbb{H} \backslash K_0$. There exist constants $\alpha_1, \alpha_2, \alpha_3 > 0$ depending on (T, G, α, κ) such that if we further define*

$$\begin{aligned} \varphi(x) &\coloneqq \alpha_1 x^{1/8} + \alpha_2 x^{1/4} + \alpha_3 x^{7/8}, \\ \zeta(x) &\coloneqq 2\frac{x^{\nu/8}}{a^{2\nu}} + 2x^{3/4} e^{-1 / 2x^{3/2}}, \end{aligned} \tag{4.23}$$

for all $x \in \mathbb{R}_+$. Then $\lim_{x \to 0^+} \varphi(x) = 0$, $\lim_{x \to 0^+} \zeta(x) = 0$ and we have

$$\mathbb{P}\bigg(\|g(t,z) - g^*(t,z)\|_{[0,T]\times G, \infty} > \varphi(\kappa^* - \kappa) \bigg) < \zeta(\kappa^* - \kappa). \tag{4.24}$$

Proof. From *Eqn.* (3.1) and *Eqn.* (3.15), we see for $k = 1, 2$ that

$$\begin{aligned} d\lambda_k(t) - d\lambda_k^*(t) &= \frac{2}{\kappa} \cdot \frac{dt}{\lambda_k(t) - \lambda_{3-k}(t)} - \frac{2}{\kappa^*} \cdot \frac{dt}{\lambda_k^*(t) - \lambda_{3-k}^*(t)} \\ &= (-1)^{3-k} \frac{2}{\kappa^*\kappa} \cdot \frac{\kappa^* X_t^* - \kappa X_t}{X_t^* X_t} dt. \end{aligned} \tag{4.25}$$

To obtain an expression of $\lambda_k(t) - \lambda_k^*(t)$, we need to express the process $\kappa^* X_t^* - \kappa X_t$. Indeed, we have

$$\kappa^* dX_t^* - \kappa dX_t = 4\bigg(\frac{1}{X_t^*} - \frac{1}{X_t}\bigg)dt + (\kappa^* - \kappa)dW_t. \tag{4.26}$$

Integrate both sides, we write

$$\kappa^* X_t^* - \kappa X_t = (\kappa^* - \kappa) \cdot a + 4\int_0^t \frac{X_s - X_s^*}{X_s^* X_s} ds + (\kappa^* - \kappa)W_t. \tag{4.27}$$

On the other hand, inspecting the above equation, we see another term $X_t^* - X_t$ appears in the integrand. Based on Lemma 3.2, we have

$$\sup_{0\le s\le t} |X_s^* - X_s| \le \frac{2\sqrt{t}}{\kappa}(\kappa^* - \kappa)^{1/2}. \tag{4.28}$$

At this moment, we have obtained an explicit form of $\kappa^* X_t^* - \kappa X_t$, which is contained in the expression of $d\lambda_k(t) - d\lambda_k^*(t)$. Define $M_t \coloneqq \inf_{0\le s\le t} X_s$ as the running infimum of the Bessel process X_t. The running infimum M_t^* of X_t^* is similarly defined. We further denote by $M_\infty = \lim_{t\to\infty} M_t$ the infimum of X_t. And similarly, we denote by $M_\infty^* = \lim_{t\to\infty} M_t^*$ the infimum of X_t^*. Indeed, from ([9] *Eqn.* 2.1) we know that

$$\begin{aligned} \mathbb{P}(M_\infty < y) &= \frac{y^{2\nu}}{a^{2\nu}} \cdot \mathbb{1}_{y\in[0,a]}, \\ \mathbb{P}(M_\infty^* < y) &= \frac{y^{2\nu^*}}{a^{2\nu^*}} \cdot \mathbb{1}_{y\in[0,a]}. \end{aligned} \tag{4.29}$$

Combining *Eqn.* (3.20) and *Eqn.* (3.26), then

$$\begin{aligned} |\lambda_k(t) - \lambda_k^*(t)| \le (\kappa^* - \kappa)\frac{2a}{\kappa^*\kappa} \cdot \frac{t}{M_t^* M_t} + (\kappa^* - \kappa)^{\frac{1}{2}} \frac{16}{\kappa^*\kappa^2} \cdot \frac{t^{\frac{5}{2}}}{(M_t^* M_t)^2} \\ + (\kappa^* - \kappa)\frac{2}{\kappa^*\kappa} \cdot \frac{t}{M_t^* M_t} \sup_{0\le s\le t} |W_s|. \end{aligned} \tag{4.30}$$

Considering $\kappa^* > \kappa$, it is then obvious that

$$\begin{aligned} \sup_{t\in[0,T]} |\lambda_k(t) - \lambda_k^*(t)| \le (\kappa^* - \kappa)\frac{2a}{\kappa^2} \cdot \frac{T}{M_\infty^* M_\infty} + (\kappa^* - \kappa)^{\frac{1}{2}} \frac{16}{\kappa^3} \cdot \frac{T^{\frac{5}{2}}}{(M_\infty^* M_\infty)^2} \\ + (\kappa^* - \kappa)\frac{2}{\kappa^2} \cdot \frac{T}{M_\infty^* M_\infty} \sup_{0\le s\le T} |W_s| \end{aligned} \tag{4.31}$$

Based on *Eqn.* (3.27), define the following events

$$\begin{aligned} E_1 &\coloneqq \left\{ M_\infty \ge (\kappa^* - \kappa)^{\frac{1}{16}} \right\}, \\ E_2 &\coloneqq \left\{ M_\infty^* \ge (\kappa^* - \kappa)^{\frac{1}{16}} \right\}, \\ E_3 &\coloneqq \left\{ \sup_{0\le s\le T} |W_s| \le \frac{1}{(\kappa^* - \kappa)^{\frac{3}{4}}} \right\}. \end{aligned} \tag{4.32}$$

Since $\kappa^* > \kappa$ and by *Eqn.* (3.27), we know that

$$\begin{aligned}\mathbb{P}\big(E_1\big) &= 1 - \frac{(\kappa^* - \kappa)^{\frac{\nu}{8}}}{a^{2\nu}},\\ \mathbb{P}\big(E_2\big) &= 1 - \frac{(\kappa^* - \kappa)^{\frac{\nu^*}{8}}}{a^{2\nu^*}}.\end{aligned} \tag{4.33}$$

From ([10], *Cor.* 2.2), we know the supremum Brownian motion S_t admits the distribution

$$\mathbb{P}\bigg(S_t \leq x\bigg) = 2\Phi\bigg(\frac{x}{\sqrt{t}}\bigg) - 1, \tag{4.34}$$

for all $x \geq 0$ and where $\frac{d}{dx}\Phi(x) \coloneqq e^{-x^2/2}/\sqrt{2\pi}$ is the law of standard normal variable. It follows from the reflection principle that

$$1 - \mathbb{P}\big(E_3\big) = 2\mathbb{P}\big(S_T \geq (\kappa^* - \kappa)^{-\frac{1}{2}}\big) \leq 2\sqrt{\frac{2}{\pi}}(\kappa^* - \kappa)^{\frac{3}{4}} \cdot e^{-\frac{1}{2(\kappa^*-\kappa)^{3/2}}}. \tag{4.35}$$

Choose $\alpha_1 \coloneqq C(T,G)\frac{4T}{\kappa^2}$, $\alpha_2 \coloneqq C(T,G)\frac{32T^{\frac{5}{2}}}{\kappa^3}$, $\alpha_3 \coloneqq C(T,G)\frac{4Ta}{\kappa^2}$. It follows from Proposition 2.3 and *Eqn.* (3.32) that on the event $E_1 \cap E_2 \cap E_3 \subset \Omega$, we have the estimate

$$\begin{aligned}\|g(t,z) - g^*(t,z)\|_{[0,T]\times G,\infty} &\leq C(T,G)\sum_{k=1}^{2}\|\lambda_k(t) - \lambda_k^*(t)\|_{[0,T],\infty}\\ &\leq \alpha_1(\kappa^* - \kappa)^{1/8} + \alpha_2(\kappa^* - \kappa)^{1/4} + \alpha_3(\kappa^* - \kappa)^{7/8}.\end{aligned} \tag{4.36}$$

On the other hand, from *Eqn.* (3.33) and *Eqn.* (3.35) we have

$$\mathbb{P}\big(E_1 \cap E_2 \cap E_3\big) \geq 1 - 2\frac{(\kappa^* - \kappa)^{\frac{\nu}{8}}}{a^{2\nu}} - 2(\kappa^* - \kappa)^{\frac{3}{4}} \cdot e^{-\frac{1}{2(\kappa^*-\kappa)^{3/2}}}, \tag{4.37}$$

where $a = a_1 - a_2 > 0$. Hence, the result is verified. □

Corollary 4.5. *Suppose $g_t(z) : \mathbb{H}\backslash K_0 \to \mathbb{H}$ is a forward Loewner chain generated by two Dyson Brownian motions $\{\lambda_1(t), \lambda_2(t)\}$ with parameter $\kappa \in (0,4)$. Suppose there is a sequence of forward Loewner chains $g_t^n(z)$ generated by Dyson Brownian motions $\{\lambda_1^n(t), \lambda_2^n(t)\}$ with parameter κ_n such that $\lim_{n\to\infty} \kappa_n = \kappa$. Then we have $g_t^n(z) \xrightarrow{Cara} g_t(z)$ for all $t \in [0,T]$, almost surely.*

5 Variant estimate on the Hausdorff distance

For any two compact sets $A, B \subset \mathbb{C}$, define the Hausdorff metric ([12] *Sec.* 6.1) by

$$d_H(A,B) \coloneqq \inf\{\epsilon > 0;\, A \subset \bigcup_{z\in B} \mathcal{B}(z,\epsilon),\, B \subset \bigcup_{z\in A} \mathcal{B}(z,\epsilon)\}, \tag{5.1}$$

where $\mathcal{B}(z,\epsilon)$ is the ϵ-ball centered at $z \in \mathbb{C}$. In this section, we prove a variant pathwise perturbation estimate in Hausdorff distance of the hulls K_t generated by the forward Loewner flow. Following [1], we have that for $\kappa \in (0,4]$, the multiple SLE curves are a.s. simple and non-intersecting. This serves as a motivation to understand the Hausdorff distance convergence following the analysis of the Caratheodory type convergence. In general, Hausdorff distance convergence is stronger than the Caratheodory convergence, however we are a.s. in the case of simple non-intersecting curves.

Going back to the N-curve case. First consider a forward Loewner chain $g_t(z) : \mathbb{H}\backslash K_t \to \mathbb{H}$ driven by forces $t \mapsto \big(\lambda_1(t), \dots, \lambda_N(t)\big)$. Denote the inverse map corresponding to $g_t(z)$ by $f_t(z) : \mathbb{H} \to \mathbb{H}\backslash K_t$, with $g_t\big(f_t(z)\big) = z$, for all $z \in \mathbb{H}$. On the other hand, consider the time-reversed forward Loewner chain generated by the time-reversed forces $t \mapsto \big(\lambda_1(T-t), \dots, \lambda_N(T-t)\big)$. Denote this forward Loewner chain by $h_t(z)$ for $z \in \mathbb{H}$. Then it satisfies

$$\partial_t h_t(z) = \frac{1}{N}\sum_{j=1}^{N} \frac{-2}{h_t(z) - \lambda_j(T-t)}, \tag{5.2}$$

with $h_0(z) = z$ for all $z \in \mathbb{H}$. Similar to the $N = 1$ case as in ([13] *Sec.* 2.), we could verify that $f_T(z) = h_T(z)$ for all $z \in \mathbb{H}$. When the system is under perturbation, we need to compare a Loewner chain with its perturbed counterpart. Indeed, denote by $f_k(t,z)$ and $g_k(t,z)$ the Loewner chains driven by $\{V_{k,1}(t), \dots, V_{k,N}(t)\}$ for $k = 1, 2$. Denote by $h_k(t,z)$ the backward Loewner chains driven by $\{V_{k,1}(T-t), \dots, V_{k,N}(T-t)\}$ for $k = 1, 2$. The following lemma estimates pathwisely the backward Loewner chain.

Lemma 5.1. *For all $\delta > 0$, there exists a constant $C(\delta, T) = \sqrt{1 + 4T/\delta^2}$ such that whenever $\operatorname{Im} z \geq \delta$, we have*

$$|h_1(T,z) - h_2(T,z)| \leq C(\delta,T) \sum_{j=1}^{N} \|V_{1,j}(T-t) - V_{2,j}(T-t)\|_{[0,T],\infty}. \tag{5.3}$$

Proof. The proof is similar to Proposition 2.3. Take $I_{k,j} = \log \frac{\operatorname{Im} h_k(t,z)}{\operatorname{Im} z}$. □

We also need the following Koebe distortion theorem, see ([14] *Lem.* 2.1)

Lemma 5.2. *Let D be a simply connected domain and assume $f : D \to \mathbb{C}$ is conformal map. Let $d = dist(z, \partial D)$ for $z \in D$. If $|z - w| \le rd$ for some $0 < r < 1$, then*

$$\frac{|f'(z)|}{(1+r)^2}|z-w| \le |f(z) - f(w)| \le \frac{|f'(z)|}{(1-r)^2}|z-w|. \tag{5.4}$$

Proposition 5.3. *Let $g_k(t,k) : [0,T] \times \mathbb{H}\backslash K_{k,t}$ be two forward Loewner chains driven by forces $t \mapsto \big(V_{k,1}(t), \dots, V_{k,N}(t)\big)$ with hulls $K_{k,t}$, for $k = 1, 2$. Let $f_k(t,z)$ be their inverse so that $g_k\big(t, f_k(t,z)\big) = z$. Write $f_k(z) := f_k(T,z)$, for $k = 1, 2$. Suppose that*

$$\sum_{j=1}^{N} \sup_{0 \le t \le T} |V_{1,j}(t) - V_{2,j}(t)| < \epsilon, \tag{5.5}$$

where $\epsilon > 0$ is taken sufficiently small. Suppose further there exists $\beta \in (0,1)$ such that for all $\zeta \in \mathbb{R}$, we have

$$|f_1'(\zeta + i\delta)| \le \delta^{-\beta}, \tag{5.6}$$

for all $\delta \le 4\sqrt{T\epsilon}$. Then, we have the Hausdorff metric estimate

$$d_H(K_{1,T} \cup \mathbb{R}, K_{2,T} \cup \mathbb{R}) \le 8(T\epsilon)^{\frac{1-\beta}{2}} + 3\sqrt{\epsilon(1+\epsilon)}. \tag{5.7}$$

Proof. Denote by $h_k(t,z)$ the time-reversed Loewner chains driven by $\{V_{k,1}(T - t), \dots, V_{k,N}(T-t)\}$ for $k = 1, 2$. Based on Lemma 4.1 and the observation that $f_k(z) = h_k(T,z)$, we know

$$|f_1(z) - f_2(z)| \le \epsilon \cdot \sqrt{1 + 4T/\delta^2}, \tag{5.8}$$

whenever $\operatorname{Im} z \ge \delta$. Take $\delta_0 = 4\sqrt{T\epsilon}$, we have

$$\sup_{\operatorname{Im} z \ge \frac{\delta_0}{2}} |f_1(z) - f_2(z)| \le \sqrt{\epsilon(1+\epsilon)}. \tag{5.9}$$

Hence, Cauchy's integral formula implies

$$\begin{aligned}\sup_{\operatorname{Im} z \geq \delta_0} |f_1'(z) - f_2'(z)| &\leq \sqrt{\epsilon(1+\epsilon)} \sup_{\operatorname{Im} z \geq \frac{\delta_0}{2}} \frac{1}{2\pi i} \oint_{\partial \mathcal{B}(z, \delta_0/2)} \frac{d\zeta}{|z-\zeta|^2} \\ &\leq \sqrt{\frac{1+\epsilon}{4T}}.\end{aligned} \tag{5.10}$$

For notational convenience, we write $\widehat{K_k} := K_{k,T} \cup \mathbb{R}$, for $k = 1, 2$. Fix $\zeta \in \mathbb{R}$, by Lemma 4.2, we have

$$\left| f_1(\zeta + i0^+) - f_1(\zeta + i\delta) \right| \leq \delta \cdot |f_1'(\zeta + i\delta)| \leq \delta^{1-\beta} \leq (16T\epsilon)^{\frac{1-\beta}{2}}. \tag{5.11}$$

Hence, we have

$$f_1\big(\{\operatorname{Im} z \leq \delta_0\}\big) \subset \bigcup_{z \in \widehat{K}_1} \mathcal{B}\big(z, (16T\epsilon)^{\frac{1-\beta}{2}}\big). \tag{5.12}$$

It is obvious that

$$\widehat{K}_2 \subset f_2\big(\{\operatorname{Im} z \leq \delta_0\}\big). \tag{5.13}$$

For the above fixed $\zeta \in \mathbb{R}$, write $w := f_1(\zeta + i0^+) \in \widehat{K}_1$. Choose $\widehat{w} \in \widehat{K}_2$ be the point in $\widehat{K}_2$ nearest to $f_2(\zeta + i\delta_0)$. Notice that this point is admissible in $\widehat{K}_2$, because $K_{2,T} \subset \mathbb{H}$ is compact. By Lemma 4.2 again

$$\begin{aligned}|\widehat{w} - f_2(\zeta + i\delta_0)| &\leq \left| f_2(\zeta + i0^+) - f_2(\zeta + i\delta_0) \right| \\ &\leq \delta_0 \cdot |f_2'(\zeta + i\delta_0)| \\ &\leq \delta_0 \cdot \Big(|f_1'(\zeta + i\delta_0)| + |f_1'(\zeta + i\delta_0) - f_2'(\zeta + i\delta_0)| \Big) \\ &\leq (16T\epsilon)^{\frac{1-\beta}{2}} + \sqrt{4\epsilon(1+\epsilon)}.\end{aligned} \tag{5.14}$$

Hence, we see that

$$\begin{aligned}|w - \widehat{w}| &\leq |w - f_1(\zeta + i\delta_0)| + |f_1(\zeta + i\delta_0) - f_2(\zeta + i\delta_0)| + |\widehat{w} - f_2(\zeta + i\delta_0)| \\ &\leq (16T\epsilon)^{\frac{1-\beta}{2}} + \sqrt{\epsilon(1+\epsilon)} + (16T\epsilon)^{\frac{1+\beta}{2}} + \sqrt{4\epsilon(1+\epsilon)} \\ &\leq 8(T\epsilon)^{\frac{1-\beta}{2}} + 3\sqrt{\epsilon(1+\epsilon)}.\end{aligned} \tag{5.15}$$

Hence the result is verified. □

Remark 5.4. *The above proposition assumes the continuity of $f_t(z)$ up to the boundary of $\mathbb{H}$, which is discussed in ([15] Thm.4.1).*

Remark 5.5. *The above proposition uniform boundedness w.r.t. the absolute value of the derivative of $f_t(z)$ on a horizontal line in $\mathbb{H}$. We expect this type of property to hold when the forward Loewner map is generated by Dyson Brownian motions. Then the Hausdorff metric can be used to estimate perturbation of the hulls K_T.*

References

[1] Makoto Katori1 and Shinji Koshida. Three Phases of Multiple SLE Driven by Non-colliding Dyson's Brownian Motions. Journal of Physics A: Mathematical and Theoretical, 2021.

[2] David Grabiner. Brownian Motion in a Weyl Chamber, Non-colliding Particles, and Random Matrices. Annales de l'Institut Henri Poincare, 1999.

[3] Martin Bender. Global Fluctuations in General β Dyson's Brownian Motion. Stochastic Processes and their Applications, 2008.

[4] Walter Rudin. Real and Complex Analysis. Higher Mathematics Series, 1966.

[5] Antti Kemppainen. Shramm-Loewner Evolution. Springer Briefs in Mathematical Physics, 2017.

[6] Henri Elad Altman. Bismut-Elworthy-Li Formulae for Bessel Processes, 2017.

[7] Carto Wong. Smoothness of Loewner Slits, 2012.

[8] Dmitry Beliaev, Terry J. Lyons, and Vlad Margarint. A New Approach to SLE Phase Transition, 2020.

[9] Zhan Shi. How Long Does It Take a Transient Bessel process To Reach Its Future Infimum. Séminaire de probabilités Strasbourg, 1996.

[10] Ben Boukai. An Explicit Expression for The Distribution of The Supremum of Brownian Motion With a Change Point. Communication in Statistics, Theory and Methods, 1990

[11] Scott Sheffield and Nike Sun. Strong Path Convergence from Loewner Driving Function Convergence. Annals of Probability, 2012.

[12] Alexey A. Tuzhilin. Lectures on Hausdorff and Gromov-Hausdorff Distance Geometry, 2020.

[13] Fredrik Johansson and Gregory F. Lawler. Optimal Holder Exponent for The SLE Path. Duke Math 2009.

[14] Christian Benes, Fredrik Johansson Viklund, and Michael J. Kozdron. On The Rate of Convergence of Loop-erased Random Walk to SLE(2). Communications in Mathematical Physics, 2013.

[15] Steffen Rohde and Oded Schramm. Basic Properties of SLE. Annals of Mathematics, 2005.

[16] Vadim Gorin and Mykhaylo Shkolnikov. Multilevel Dyson Brownian Motions via Jack Polynomials. Probability Theory and Related Fields, 2015.

[17] Gregory F. Lawler. Conformally Invariant Processes in the Plane.Mathematical Surveys and Monographs, 2005.

[18] John. Cardy. Stochastic Loewner evolution and Dyson's circular ensembles, Journal of Physics A: Mathematical and General, 2003

[19] Katori, Makoto and Koshida, Shinji. Conformal welding problem, flow line problem, and multiple Schramm–Loewner evolution,

[20] Andrea del Monaco and Schleissinger, Sebastian: Multiple SLE and the complex Burgers equation, 2016

[21] J. Lenells, F. Viklund. Schramm's formula and Green's function for multiple SLE, 2017.

[22] Vincent Beffara, Eveliina Peltola, and Hao Wu. On the uniqueness of global multiple SLEs, 2018.

[23] Eveliina Peltola and Hao Wu. Global and Local Multiple SLEs for $\kappa \leq 4$ and Connection Probabilities for Level Lines of GFF . *Communications in Mathematical Physics*. 2019.

[24] V. Healey and G. Lawler. N-sided Radial Schramm-Loewner Evolution. 2019

[25] A. Karrila. Multiple SLE type scaling limits: from local to global. 2019

[26] A. Karrila. UST branches, martingales, and multiple SLE(2).2020