Convergence of Splitting Schemes to Schramm-Loewner Evolutions

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Abstract

In the first part of the paper we study the approximation of the SLE_κ traces via the Ninomiya-Victoir Splitting Scheme. We prove a strong convergence in probability *w.r.t.* the *sup*-norm to the distance between the SLE trace and the output of the Ninomiya-Victoir Splitting Scheme when applied in the context of the Loewner Differential Equation. We show also that an L^p convergence of the scheme can be achieved under a set of assumptions. In the last section we show the uniform convergence of the approximation of the SLE trace obtained using a different scheme that is based on the linear interpolation of the Brownian driver.

1 Introduction

The Loewner equation was introduced by Charles Loewner in 1923 in and it was one of the important ingredients in the proof of the Bieberbach Conjecture that was done by Louis de Branges, years later in 1985. In 2000, Oded Schramm introduced a stochastic version of the Loewner equation. The stochastic version of the Loewner evolution, i.e. the Schramm-Loewner evolution, SLE_κ, generates a one parameter family of random fractal curves that are proved to describe scaling limits of a number of discrete models that appear in two-dimensional statistical physics. For example, in ([17], *Sec.* 1.1) it is shown that the scaling limit of loop erased random walk, with the loops erased in a chronological order, converges in the scaling limit to SLE_κ with $\kappa = 2$. Moreover, other two dimensional discrete models from statistical mechanics including Ising model cluster boundaries, Gaussian free field interfaces, percolation on the triangular lattice at critical probability, and uniform spanning trees were proved to converge in the scaling limit to SLE_κ for values of $\kappa = 3$, $\kappa = 4$, $\kappa = 6$, and $\kappa = 8$ respectively in the series of works [20], [21], [22[, [23].

There are various versions of Loewner equations. One of them is the forward Loewner equations defined in the upper half-plane $\mathbb{H}$ and a fixed time interval $[0,T]$

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \tag{1.1}$$

with the initial condition $g_0(z) = z$, for all $z \in \mathbb{H}$ and the continuous driving force $\lambda : [0,T] \to \mathbb{R}$.

The family of maps $(g_t)_{0 \le t \le T}$ is called the forward Loewner chain. For all $z \in \mathbb{H}$, the solution of the above forward Loewner equation is uniquely defined up to $T_z = \inf\{t \ge 0,\, g_t(z) = \lambda(t)\}$. Over time, the hulls, that is the sets $K_t = \{z \in \mathbb{H},\, T_z \le t\}$ grow. It is also known that for all $t \in [0,T]$, there is a unique conformal map $g_t : \mathbb{H} \backslash K_t \to \mathbb{H}$ satisfying the hydrodynamic normalization

$$\lim_{z \to \infty} [g_t(z) - z] = 0. \tag{1.2}$$

We study $(g_t)_{0 \le t \le T}$ parametrized by upper half-plane capacity

$$g_t(z) = z + \frac{2t}{z} + o(1/|z|),\ as\ |z| \to \infty, \tag{1.3}$$

where by ([1], *Lem.* 4.1), the coefficient of the z-term is 1 and each coefficient a_k of the term z^{-k}, $k \in \mathbb{N}_+$ is real.

We are particularly interested in the case when the Loewner chain $(g_t)_{0 \le t \le T}$ is generated by a curve $\gamma : [0,T] \to \mathbb{H} \cup \{\lambda(T)\}$. By ([2], *Thm.* 4.1), this is equivalent to the existence and continuity in $0 < t < T$ of

$$\gamma(t) = \lim_{\epsilon \to 0^+} g_t^{-1}(\lambda(t) + i\epsilon). \tag{1.4}$$

Then, for each $t \in [0,T]$. the domain of the map $g_t(z)$ is the unbounded component of $\mathbb{H} \backslash \gamma([0,t])$.

Throughout our paper, we work with stochastic Loewner chains and for this we introduce a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the Brownian motion is defined. Furthermore, for $\kappa \in \mathbb{R}_+$, we consider the forward Loewner chain driven by Brownian motion $\sqrt{\kappa} B_t$, with $t \in [0,1]$

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \tag{1.5}$$

with $g_0(z) = z$. It was shown in ([2] *Sec.* 5) that the Loewner chains driven by Brownian motion are generated by a trace for $\kappa \neq 8$, that is the following limit exists and is continuous in time a.s.

$$\gamma(t) = \lim_{y \to 0^+} g_t^{-1}(\sqrt{\kappa} B_t + iy) = \lim_{y \to 0^+} \widehat{g}_t^{-1}(iy). \tag{1.6}$$

We call $\gamma(t)$ the SLE_κ trace.

Throughout the proof we will use the backward Loewner equation which is related to the forward version $Eqn.$ (2.1) and admits the form

$$\partial_t h_t(z) = \frac{-2}{h_t(z) - \sqrt{\kappa} B_{1-t}}, \tag{1.7}$$

with $h_0(z) = z$. This backward equation generates a Loewner curve $\eta : [0,1] \to \mathbb{H} \cup \{0\}$. Notice that $g_t(z)$ and $h_t(z)$ are both driven by Brownian motions with different time directions. In fact, there is a correspondence ([13], $Sec.$ 1.1) between the two evolutions. The random set $\eta([0,1])$ has the same law as the chordal SLE_κ trace $\gamma([0,1])$ modulo a real scalar shift $\sqrt{\kappa} B_{T=1}(\omega)$ for $\omega \in \Omega$. For convenience, we call $\eta(t)$ the shifted Loewner curve. We will simulate the Loewner curve $\eta(t)$ instead of $\gamma(t)$ for convenience, since the former curve preserves statistical properties of the latter curve.

In the last years, results on numerical schemes to approximate the SLE_κ traces, along with mathematical proofs of their convergence, appeared in the body of literature on the topic. For example, V. Beffara used a Euler Scheme to produce approximations of the SLE_κ hulls. For other methods of approximations of the SLE traces, we refer the reader to [3],[11],[18],[19].

In this paper, we study two schemes to simulate the SLE trace and we provide theoretical bounds of their convergence. The first one is the Ninomiya-Victoir and the second one is obtained from the linear-interpolation of the Brownian driver.

The paper is divided in several sections, the first one being the introduction. In the second section we introduce the Ninomiya-Victoir Splitting Scheme. In the third section we prove the strong convergence in probability of the Ninomiya-Victoir Splitting Scheme, and in the next section, we show the L^p convergence of this scheme under a set of assumptions. In the last section, we discuss the linear-interpolation of the Brownian driver approximation scheme.

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2 Ninomiya-Victoir Splitting Scheme

In this section, we introduce the Ninomiya-Victoir Splitting Scheme. Before that, we will rephrase the forward and backward Loewner evolutions in a convenient manner. If we set $\widehat{g}_t(z) := g_t(z) - \sqrt{\kappa}B_t$ for all $\in \mathbb{H}\backslash K_t$, then the forward Loewner chain driven by Brownian motion can be rewritten as

$$\begin{aligned} d\widehat{g}_t(z) &= \frac{2}{\widehat{g}_t(z)}dt - \sqrt{\kappa}dB_t, \\ \widehat{g}_0(z) &= z, \end{aligned} \tag{2.1}$$

with $z \in \mathbb{H}\backslash K_t$.

Moreover, let us consider $Z_t(z) := h_t(z) - \sqrt{\kappa}B_t$, for all $z \in \mathbb{H}$ (see [3], *Eqn.* 6.5). Then, the backward Loewner differential equation driven by Brownian motion can be rewritten as

$$\begin{aligned} dZ_t &= -\frac{2}{Z_t}dt + \sqrt{\kappa}dB_t, \\ Z_0 &= iy, \end{aligned} \tag{2.2}$$

For the initial condition, we consider $y > 0$ to be taken sufficiently small.

We are now ready to introduce the Ninomiya-Victoir Scheme.

Definition 2.1. ***Ninomiya-Victoir Scheme***
Consider n-dimensional SDE on $\mathbb{R}_+$ with the form

$$\begin{aligned} dW_t &= L_0(W_t)dt + L_1(W_t)dB_t, \\ W_0 &= \xi, \end{aligned} \tag{2.3}$$

where $\xi \in \mathbb{R}^n$ and $L_i : \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector fields. For all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$, let the flow $\exp(tL_i)x$, $i = 1, 2$, denote the unique solution at time $u = 1$ to the ODE

$$\begin{aligned} \frac{dy}{du} &= tL_i(y), \\ y(0) &= x. \end{aligned} \tag{2.4}$$

For a fixed iteration step $n \in \mathbb{N}$, we choose an arbitrary, possibly non-uniform, partition $\{t_0 = 0, t_1, \ldots, t_n = 1\}$ with step-size $h_k = t_{k+1} - t_k$. We approximate a numerical solution $\{\widetilde{W}_{t_k}\}_{0 \le k \le n}$ in the sense that $\widetilde{W}_0 = \xi$ and

$$\widetilde{W}_{t_{k+1}} = \exp\left(\frac{1}{2}h_k L_0\right) \exp\left(B_{t_k,t_{k+1}} L_1\right) \exp\left(\frac{1}{2}h_k L_0\right)\widetilde{W}_{t_k}, \tag{2.5}$$

for all $k = 0, 1, \ldots, n$ and where $B_{t_k,t_{k+1}}$ is the abbreviation for $B_{t_{k+1}} - B_{t_k}$. In *fact, the approximation* $\{\widetilde{W}_{t_k}\}_{0\le k\le n}$ *enjoys an integral form between every two discretization points*

$$\widetilde{W}_t = \xi + \frac{1}{2}\int_0^t L_0(\widetilde{W}_s^{(2)})ds + \int_0^t L_1(\widetilde{W}_s^{(1)})dB_s + \frac{1}{2}\int_0^t L_0(\widetilde{W}_s^{(0)})ds, \tag{2.6}$$

where the above three discretization processes defined on each time interval $[t_k, t_{k+1}]$ *admit the form*

$$\begin{aligned} \widetilde{W}_t^{(0)} &:= \exp\left(\frac{1}{2}(t-t_k)L_0\right)\widetilde{W}_{t_k}, \\ \widetilde{W}_t^{(1)} &:= \exp\left(B_{t_k,t}L_1\right)\widetilde{W}_{t_{k+1}}^{(0)}, \\ \widetilde{W}_t^{(2)} &:= \exp\left(\frac{1}{2}(t-t_k)L_0\right)\widetilde{W}_{t_{k+1}}^{(1)}. \end{aligned} \tag{2.7}$$

Looking back at our backward Loewner equation, we write $L_0(z) = -2/z$ and $L_1(z) = \sqrt{\kappa}$. Hence, by ([3], $Thm.$ 6.2) the following form is immediate

$$\begin{aligned} \exp(tL_0)z &= \sqrt{z^2 - 4t}, \\ \exp(tL_1)z &= z + \sqrt{\kappa}t. \end{aligned} \tag{2.8}$$

Given the above arbitrary partition $\{t_0 = 0, t_1, \ldots, t_n = 1\}$, we could formulate an approximated solution $\{\widetilde{Z}_{t_k(t)}\}_{0\le k\le n}$ via the Ninomiya-Victoir Splitting Scheme

$$\widetilde{Z}_{t_{k+1}} := \sqrt{\left(\sqrt{\widetilde{Z}_{t_k}^2 - 2h_k} + \sqrt{\kappa}B_{t_k,t_{k+1}}\right)^2 - 2h_k}\,, \tag{2.9}$$

with the initial value $\widetilde{Z}_0 = iy$ specified at each n^{th} iteration.

3 Strong convergence in probability

In the following, we use $\|\cdot\|_{[0,1],\infty}$ for the sup-norm on the interval $[0,1]$. In addition, we use $||\cdot||$ to denote the mesh size of our partition of the time interval. In this section we give a strong convergence in probability to the decay rate of the $\|\cdot\|_{[0,1],\infty}$ norm (*i.e. supremum norm*) between the original Loewner curve and our scheme.

Definition 3.1. *Let* $Z_t(iy_n)$ *be the solution to Eqn.* (2.5) *started from* $iy_n \in \mathbb{H}$, *and* $\widetilde{Z}_t(iy_n)$ *be its approximation following Ninomiya-Victoir Scheme, and let* $\eta(t)$ *be the shifted Loewner curve defined before.*

Notice that at each iteration step $n \in \mathbb{N}$, we specify an initial condition

$y \in \mathbb{R}_+$ and let the approximated sample paths $(\widetilde{Z}_t(iy))_{0\le t\le 1}$ evolve according to the backward Loewner equation (2.5). To ensure a $\|\cdot\|_{[0,1],\infty}$ convergence result, we do not only require the mesh of the partition tends to 0, but also choose a sequence $\{y_n\} \subset \mathbb{R}_+$ so that $y_n \to 0^+$ strictly monotonically.

Remark 3.2. *Notice that we cannot let $y_n \equiv y$ for some $y > 0$, otherwise the convergence pattern breaks down and hence strict monotonicity of $\{y_n\}$ is necessary. On the other hand, the decay rate of $\{y_n\}$ should not be too fast to destroy the probability inequality w.r.t. the $\|\cdot\|_{[0,1],\infty}$ norm, which we will see in the following context.*

In this section, we manually set $y_n = n^{-1/2}$ for all $n \in \mathbb{N}$. This choice of $\{y_n\}$ actually satisfies the requirements in the above *Rmk*. 3.2 for the initial conditions. We consider $\mathcal{D}_n$ to be a uniform partition of $[0,1]$ with mesh-size $||\mathcal{D}_n||$.

Definition 3.3. *For all $t \in [0,1]$, given an arbitrary uniform partition $\mathcal{D}_n$, we define $\{t_k(t), t_{k+1}(t)\} \subset \mathcal{D}_n$ to be the neighboring two points in the partition $\mathcal{D}_n$ between which t resides, i.e. $t_k(t) \le t < t_{k+1}(t)$.*

We emphasize that the index k is changing as the parameter t evolves in the time interval. We preferred this notation to make it close to the notation used in the definition of the splitting scheme.

We are now reaching our main object: an upper bound for the probability of $\|\cdot\|_{[0,1],\infty}$ norm of $(\eta(t) - \widetilde{Z}_t(iy_n))_{0\le t\le 1}$ to be small in the sense given by the following theorem.

Theorem 3.4. *Let $\eta(t)$ be the backward SLE trace for $\kappa \neq 8$. There exist two non-increasing functions $\varphi_i : \mathbb{Z}_+ \to \mathbb{R}_+$ such that $\lim_{n\to\infty} \varphi_i(n) = 0^+$ with $i = 1, 2$. If the mesh $||D_n|| \to 0^+$ with $n \to \infty$, faster than a proper rate $||D_n|| = o(n^{-3})$, then*

$$\mathbb{P}\left(\left\|\eta(t) - \widetilde{Z}_t(iy_n)\right\|_{[0,1],\infty} \le \varphi_1(n)\right) \ge 1 - \varphi_2(n). \tag{3.1}$$

To prove this theorem, we use

$$\begin{aligned}\left|Z_t(iy_n) - \widetilde{Z}_t(iy_n)\right| &\le \left|Z_t(iy_n) - Z_{t_k(t)}(iy_n)\right| \\ &\quad + \left|Z_{t_k(t)}(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n)\right| + \left|\widetilde{Z}_{t_k(t)}(iy_n) - \widetilde{Z}_t(iy_n)\right|,\end{aligned} \tag{3.2a}$$

in order to estimate

$$\left|\eta(t) - \widetilde{Z}_t(iy_n)\right| \le |\eta(t) - Z_t(iy_n)| + \left|Z_t(iy_n) - \widetilde{Z}_t(iy_n)\right|. \tag{3.2b}$$

The *Ineq.* (3.3a) follows from the lemma below.

Lemma 3.5. ([4], *Thm.* 3.4.2) *There exist* $c_1, c_2 > 0$ *such that if we consider the event*

$$E'_{n,1} := \left\{ osc(\sqrt{\kappa}B_t, \frac{1}{n}) \leq c_1 \sqrt{\frac{\log(n)}{n}} \right\}, \tag{3.3}$$

then we have

$$\mathbb{P}\big(E'_{n,1}\big) \geq 1 - \frac{c_2}{n^2}. \tag{3.4}$$

Lemma 3.6. ([11], *Eqn.* 21.) *There exist* $c_3, c_4 > 0$ *and* $\beta_1 \in (0,1)$ *such that if we consider the event*

$$E''_{n,1} := \left\{ \left|\partial_z \widehat{g_t}^{-1}(iv)\right| \leq c_3 \cdot v^{-\beta_1} \; for\ all\, t \in [0,1] \; and\, v \in [0, \frac{1}{\sqrt{n}}] \right\}, \tag{3.5}$$

then we have

$$\mathbb{P}\big(E''_{n,1}\big) \geq 1 - \frac{c_4}{n^{c_3/2}}. \tag{3.6}$$

We have an estimate to the first term to *Ineq.* (3.3a) with the form

$$\begin{aligned} \big|Z_t(iy_n) - Z_{t_k(t)}(iy_n)\big| \leq |Z_t(iy_n) - \eta(t)| + \big|Z_{t_k(t)}(iy_n) - \eta\big(t_k(t)\big)\big| \\ + \big|\eta(t) - \eta\big(t_k(t)\big)\big|. \end{aligned} \tag{3.7}$$

To proceed our discussion, we remind our readers of the following definition.

Definition 3.7. *A continuous function* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is called a subpower function if it is non-increasing and satisfies*

$$\lim_{x\to\infty} x^{-\nu}\phi(x) = 0, \; for\ all\, \nu > 0. \tag{3.8}$$

Remark 3.8. *A typical subpower function is* $\phi(x) = (\log x)^\alpha$, *for real* $\alpha > 0$.

With the notion of a subpower function, we have the following result.

Proposition 3.9. *Let* $\beta_1 \in (0,1)$. *There exists a subpower function* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that if we consider the event*

$$E^*_{n,1} := \left\{ \big\|\eta(t) - \eta\big(t_k(t)\big)\big\|_{[0,1],\infty} \leq \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}} \right\}, \tag{3.9}$$

and if $||\mathcal{D}_n|| \leq n^{-1}$, *then*

$$\mathbb{P}\big(E^*_{n,1}\big) \geq 1 - \frac{c_2}{n^2} - \frac{c_4}{n^{c_3/2}}. \tag{3.10}$$

Proof. In the proof we omit the bracket in $t_k(t)$ and simply write this term as $t_k(t)$, which will be clear from the context. In the proof, we follow the statement in ([11], *Lem.* 2.5) with some obvious changes of notations. Since $\eta([0,1])$ has identical distribution to $\gamma([0,1])$ modulo a scalar shift $\sqrt{\kappa}B_1$, it is immediate that $\mathbb{P}\big(E^*_{n,1}\big)$ is equal to the probability of the event with an expression which we substitute $\gamma(t)$ (*and resp.* $\gamma\big(t_k(t)\big)$) into $\eta(t)$ (*and resp.* $\eta\big(t_k(t)\big)$). By ([11], *Lem.* 2.5) there exists a subpower function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, on the event $E'_{n,1} \cap E''_{n,1} \subset \Omega$, provided $0 \le t - t_k(t) \le n^{-1}$ for all $t \in [0,1]$, we have

$$
\begin{aligned}
|\gamma(t) - \gamma(t_k(t))| &\le \phi(\sqrt{n})\bigg(\int_0^{n^{-1/2}} \big|\partial_z \widehat{g}_t^{-1}(ir)\big| dr + \int_0^{n^{-1/2}} \big|\partial_z \widehat{g}_t^{-1}(ir)\big| dr\bigg) \\
&\le \phi(n) \cdot \frac{2}{1-\beta_1} n^{-(1-\beta_1)/2},
\end{aligned}
\tag{3.11}
$$

where $\beta_1 \in (0,1)$. Hence $\mathbb{P}\big(E^*_{n,1}\big) \ge \mathbb{P}\big(E'_{n,1} \cap E''_{n,1}\big)$ and the conclusion follows. □

To finish the evaluation of *Eqn.* (3.8) and then finishing the first term in *Ineq.* (3.3*a*), we have the following result

Proposition 3.10. *Let* $\epsilon_0 \in (0,1)$*. If we choose* $M_n = n^{(1-\epsilon_0)/4}$ *and consider the event*

$$
E^{**}_{n,1} := \bigg\{ \|Z_t(iy_n) - \eta(t)\|_{[0,1],\infty} \le M_n \cdot y_n^{1-\epsilon_0} = \frac{1}{n^{(1-\epsilon_0)/4}} \bigg\}, \tag{3.12}
$$

then there exists $\epsilon_n \to 0^+$ *monotonically such that*

$$
\mathbb{P}\big(E^{**}_{n,1}\big) \ge 1 - \epsilon_n. \tag{3.13}
$$

Proof. It is stated in ([3], *Lem.* 6.7) that there exists $\epsilon_0 \in (0,1)$ so that *almost surely*, we have

$$
\sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| \le C'_1(\omega) \cdot y_n^{1-\epsilon_0}, \tag{3.14}
$$

where $C'_1(\omega)$ is *almost surely* finite. Guaranteed with the existence of at least one $C'_1(\omega) \in \mathbb{R}_+$ for almost all $\omega \in \Omega$, we define the collection $\mathcal{A}(\omega) \subset \mathbb{R}_+$ for those $\omega \in \Omega$ with which there exists at least one $C'_1(\omega)$ satisfying *Ineq.* (3.15). Notice that the collection $\mathcal{A}(\omega)$ is defined except for a *measure-zero* event. The well-ordering principle tells us that $\mathcal{A}(\omega)$ has a lower bound. Hence it is legitimate to define

$$
C_1(\omega) := \inf \mathcal{A}(\omega), \tag{3.15}
$$

which is *almost surely* defined. Hence, we could simply assume $C(\omega)$ exists and is finite everywhere via subtracting a *measure-zero* event from Ω. With our choice of $M_n \to \infty$, there exists $\epsilon_n \in [0,1]$ with $\epsilon_n := \min\{\epsilon_1, \dots, \epsilon_{n-1}, \mathbb{P}\big((E_{n,1}^{**})^c\big)\}$ such that

$$\mathbb{P}\left(\sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| \le M_n \cdot y_n^{1-\epsilon_0}\right) \ge 1 - \epsilon_n. \tag{3.16}$$

On the event $E_{n,1}^{**} \subset \Omega$, we know then

$$\sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| \le C_1(\omega) \cdot y_n^{1-\epsilon_0} \text{ and } \sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| \le M_n \cdot y_n^{1-\epsilon_0}. \tag{3.17}$$

By the definition of $C_1(\omega)$, it is then clear that on the event $E_{n,1}^{**}$, we have

$$C_1(\omega) \le M_n. \tag{3.18}$$

Hence, on the event $E_{n,1}^{**}$, it is immediate that

$$\sup_{t\in[0,1]} |Z_t(iy_{n+1}) - \eta(t)| \le C_1(\omega) \cdot y_{n+1}^{1-\epsilon_0} \le M_n \cdot y_{n+1}^{1-\epsilon_0} \le M_{n+1} \cdot y_{n+1}^{1-\epsilon_0}. \tag{3.19}$$

Hence the event $E_{n+1,1}^{**}$ occurs and

$$\mathbb{P}\big(E_{n+1,1}^{**}\big) \ge \mathbb{P}\big(E_{n,1}^{**}\big), \tag{3.20}$$

which justifies our choice of definition of ϵ_n, from which the monotonicity of $\{\epsilon_n\} \subset \mathbb{R}_+$ is easily seen. To show that $\epsilon_n \to 0^+$, we notice that $C_1(\omega)$ is *almost surely* finite. Suppose $\epsilon_n \to \sigma > 0$. Then with probability σ, the constant $C_1(\omega)$ is greater than any M_n, $n \in \mathbb{N}_+$. Since $M_n \to \infty$, we are forced to conclude that $C_1(\omega) = \infty$ with positive probability, which is impossible. □

We have now discussed every term in *Eqn.* (3.8), it is time to finalize the estimate of the first term in *Ineq.* (3.3*a*).

Proposition 3.11. *Let $\beta_1 \in (0,1)$. Given the assumptions that $||\mathcal{D}_n|| \le n^{-1}$ and $y_n = n^{-1/2}$, if we define the event*

$$E_{n,1} := \left\{ \big\|Z_t(iy_n) - Z_{t_k(t)}(iy_n)\big\|_{[0,1],\infty} \le \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}} + \frac{2}{n^{(1-\epsilon_0)/4}} \right\}, \tag{3.21}$$

then the following inequality holds

$$\mathbb{P}\big(E_{n,1}\big) \ge 1 - \frac{c_2}{n^2} - \frac{c_4}{n^{c_3/2}} - 2\epsilon_n. \tag{3.22}$$

Proof. On the event $E^*_{n,1} \cap E^{**}_{n,1}$, we know that for $\beta_1 \in (0,1)$, we have

$$\begin{aligned} \sup_{t\in[0,1]} |Z_t(iy_n) - \eta(t)| &\le \frac{1}{n^{(1-\epsilon_0)/4}}, \\ \sup_{t\in[0,1]} \big|\eta(t) - \eta\big(t_k(t)\big)\big| &\le \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}}. \end{aligned} \tag{3.23}$$

Looking back to *Eqn.* (3.8), we see $\mathbb{P}\big(E_{n,1}\big) \ge \mathbb{P}\big(E^*_{n,1} \cap E^{**}_{n,1}\big) \ge 1 - c_2 n^{-2} - c_4 n^{-c_3/2} - 2\epsilon_n$. □

Hence we have estimated the $\|\cdot\|_{[0,1],\infty}$ norm of the first term in *Ineq.* (3.3*a*). This is in fact the most complicated term among these three terms. Next, we will estimate the $\|\cdot\|_{[0,1],\infty}$ norm of the second term.

Inspecting *Eqn.* (2.12), we observe that the evolution $\widetilde{Z}_{t_k} \mapsto \widetilde{Z}_{t_{k+1}}$ resembles a Loewner map driven by constant forces on the sub-interval $[t_k(t), t_{k+1}]$. In fact, this is the case. We are going to split the total time interval $[0,1]$ into time sub-intervals $[t_k(t), t_{k+1}]$. On each time sub-interval, the evolution $\widetilde{Z}_{t_k} \mapsto \widetilde{Z}_{t_{k+1}}$ is a composition of two local backward Loewner maps driven by constant forces with an intermediate parallel translation.

Lemma 3.12. ([10], *Sec.* 2.) *Given a constant driving force $t \mapsto A$ on the time interval $[0,T]$, the forward Loewner chain admits the form*

$$g_t(z) = A + \big[(z-A)^2 + 4t\big]^{\frac{1}{2}}. \tag{3.24}$$

And this forward Loewner chain induces a time-reversed (i.e. backward) Loewner chain at the final moment $t = T$ with the form

$$h_T(z) = A + \big[(z-A)^2 - 4T\big]^{\frac{1}{2}}. \tag{3.25}$$

Proof. We know $g_T(z) \circ h_T(z) = z$, for all $z \in \mathbb{H}$ by ([1], *Lem.* 4.10). The result immediately follows. □

Lemma 3.13. *On each time sub-interval $[t_k(t), t_{k+1}]$, we consider the constant force $t \mapsto 0$ on time interval $[t_k(t), t_k(t) + \frac{h_k}{2}]$ and the constant force given by the corresponding value of the Brownian path at the end of the time sub-interval on $[t_k(t) + \frac{h_k}{2}, t_{k+1}]$. We denote the backward Loewner chain driven by these constant forces as $\iota'_{k,1}$ and $\iota''_{k,1}$, respectively. Consider the parallel translation $z \xmapsto{\iota_{k,2}} z + \sqrt{\kappa} B_{t_k(t),t_{k+1}}$. Then we have the composition*

$$\widetilde{Z}_{t_{k+1}}(iy_n) = \iota''_{k,1} \circ \iota_{k,2} \circ \iota'_{k,1} \widetilde{Z}_{t_k}(iy_n). \tag{3.26}$$

Proof. Inspect *Eqn* (2.12) and *Eqn.* (3.26) and the conclusion follows. □

We have the following proposition.

Proposition 3.14. *If we consider the perturbation event*

$$E_{n,2} \coloneqq \left\{ \left\| Z_{t_k(t)}(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n) \right\|_{[0,1],\infty} \leq \frac{1}{\sqrt{4n+1}} \right\}, \tag{3.27}$$

and if we further restrict $||\mathcal{D}_n|| \leq n^{-1} \wedge (4n+1)^{-3}$, *then*

$$\mathbb{P}\big(E_{n,2}\big) \geq 1 - 2e^{-(4n+1)/\kappa}. \tag{3.28}$$

Proof. From *Sec.* 2. we already know $Z_t(iy_n) = h_t(iy_n) - \sqrt{\kappa}B_t$. If we further choose $\widehat{B}_t \coloneqq B_{1-t} - B_1$, then *Eqn.* (2.2) can be written as

$$\partial_t h_t(z) = \frac{-2}{h_t(z) - \sqrt{\kappa}\widehat{B}_t}, \tag{3.29}$$

with $h_0(z) = z$ and where $\widehat{B}_t$ has the law of a standard Brownian motion. We also comment that our splitting scheme could be formulated in a similar fashion. Define the driver

$$\widetilde{\lambda}(t) \coloneqq 0 \cdot \mathbb{1}_{[0,\frac{t_1}{2})} + \sum_{k\geq 1} \sqrt{\kappa} B_{t_k(t)} \cdot \mathbb{1}_{[t_k(t)-\frac{h_k}{2}, t_k(t)+\frac{h_k}{2}\wedge 1)}. \tag{3.30}$$

The random process $\widetilde{\lambda}(t)$ can be viewed as a step-function interpolation to the sample paths of Brownian motion $\sqrt{\kappa}B_t$ on $[0,1]$. In this regard, we denote by $\widetilde{Z}^*_t(iy_n)$ the trajectory driven by the above driver similar to $Z_t(iy_n)$ being driven by $\sqrt{\kappa}\widehat{B}_t$ in the following sense

$$\widetilde{Z}^*_t(iy_n) = \widetilde{h}_t(iy_n) - \widetilde{\lambda}(t), \tag{3.31}$$

due to *Lem.* 3.13 and where $(\widetilde{h}_t)_{t\in[0,1]}$ is a backward Loewner chain constrained by

$$\partial_t \widetilde{h}_t(z) = \frac{-2}{\widetilde{h}_t(z) - \widehat{\lambda}_t}, \tag{3.32}$$

with $\widetilde{h}_0(z) = z$ and $\widehat{\lambda}_t \coloneqq \widetilde{\lambda}(1-t) - \widetilde{\lambda}(1)$. The above scheme brings us some consistency to some perturbation term $Z_t(iy_n) - \widetilde{Z}^*_t(iy_n)$. And our first goal is to estimate

$$\left| Z_t(iy_n) - \widetilde{Z}^*_t(iy_n) \right| \leq \left| h_t(iy_n) - \widetilde{h}_t(iy_n) \right| + \left| \sqrt{\kappa}B_t - \widetilde{\lambda}(t) \right|. \tag{3.33}$$

Define $\epsilon := \sup_{t\in[0,1]} \left|\sqrt{\kappa}B_t - \widetilde{\lambda}(t)\right|$, then it follows that

$$\left|Z_t(iy_n) - \widetilde{Z}_t^*(iy_n)\right| \le \left|h_t(iy_n) - \widetilde{h}_t(iy_n)\right| + \epsilon. \tag{3.34}$$

To achieve this goal, we further define $H(t) := h_t(iy_n) - \widetilde{h}_t(iy_n)$. And we will first estimate $|H(t)|$. Differentiate $H(t)$ *w.r.t.* $t \in [0,1]$ and use *Eqn.* (2.2) and *Eqn.* (3.33) to obtain

$$\frac{d}{dt}H(t) - H(t)\zeta(t) = (\sqrt{\kappa}\widehat{B}_t - \widehat{\lambda}_t)\zeta(t), \tag{3.35}$$

where we define $\zeta(t) := \big(h_t(iy_n) - \sqrt{\kappa}\widehat{B}_t\big)^{-1} \cdot \big(\widetilde{h}_t(iy_n) - \widehat{\lambda}_t\big)^{-1}$. Notice that the derivative of $H(t)$ *w.r.t.* time t is defined except for finitely many points because the driving force $\widetilde{\lambda}(t)$ is piecewise continuous. Integrating the above differential equation and choose $u(t) := e^{-\int_0^t \zeta(s)ds}$, we find

$$H(t) = u(t)^{-1}\left[H(0) - \int_0^t (\sqrt{\kappa}\widehat{B}_s - \widehat{\lambda}_s)u(s)\zeta(s)ds\right]. \tag{3.36}$$

Since $H(0) = 0$, we obtain

$$|H(t)| \le \int_0^t \left|\sqrt{\kappa}\widehat{B}_s - \widehat{\lambda}_s\right| e^{\int_s^t \operatorname{Re}\zeta(r)dr}|\zeta(s)|ds. \tag{3.37}$$

Then, it is immediate that

$$\begin{aligned} \left|h_t(iy_n) - \widetilde{h}_t(iy_n)\right| &\le \epsilon \cdot \int_0^t e^{\int_s^t \operatorname{Re}\zeta(r)dr}|\zeta(s)|ds \\ &\le \epsilon \cdot \left(e^{\int_s^t |\zeta(r)|dr} - 1\right), \end{aligned} \tag{3.38}$$

where the last inequality is due to ([15], *Lem.* 2.3) and ([15], *Eqn.* 2.12). Now turning attention to *Eqn.* (3.34), we have

$$\left|Z_t(iy_n) - \widetilde{Z}_t^*(iy_n)\right| \le \left|h_t(iy_n) - \widetilde{h}_t(iy_n)\right| + \epsilon \le \epsilon \cdot e^{\int_s^t \operatorname{Re}\zeta(r)dr}. \tag{3.39}$$

Furthermore, ([15], *Eqn.* 2.12) tells us that $\int_0^t |\zeta(s)|ds \le \log\Big(\sqrt{4+y_n^2}/y_n\Big)$. Consequently, we have

$$\left|Z_t(iy_n) - \widetilde{Z}_t^*(iy_n)\right| \le \epsilon \cdot \sqrt{4+y_n^2}/y_n = \epsilon \cdot \sqrt{4n+1}. \tag{3.40}$$

Notice that

$$\epsilon = \sup_{t\in[0,1]} \left|\sqrt{\kappa}B_t - \widetilde{\lambda}(t)\right| \leq \bigvee_{t_k(t)\in\mathcal{D}_n} \sup_{t\in[0,h_k]} \sqrt{\kappa}|B_t|, \tag{3.41}$$

where the notation "$\vee$" indicates we take the maximal value over all $t_k(t) \in \mathcal{D}_n$. By ([9], *Cor.* 2.2), for the supremum Brownian motion $S_t := \sup_{0\leq s\leq t} B_s$ we have that

$$\mathbb{P}\left(S_t \leq x\right) = 2\Phi\left(\frac{x}{\sqrt{t}}\right) - 1, \tag{3.42}$$

for all $x \geq 0$ and where $\frac{d}{dx}\Phi(x) := e^{-x^2/2}/\sqrt{2\pi}$ is the law of standard normal variable. It follows from the reflection principle that

$$\mathbb{P}\left(\sup_{0\leq t\leq h_k} \left|\sqrt{\kappa}B_t\right| \geq \frac{1}{4n+1}\right) = 2\mathbb{P}\left(S_{h_k} \geq \frac{1}{\sqrt{\kappa}\cdot(4n+1)}\right) \leq 2\sqrt{\frac{2}{\pi}}e^{-\frac{(4n+1)^{-2}}{2h_k\cdot\kappa}}, \tag{3.43}$$

if we restrict $h_k \leq n^{-1} \wedge (4n+1)^{-3}$ for all $t_k(t) \in \mathcal{D}_n$. Then

$$\left\{\epsilon > \frac{1}{4n+1}\right\} \leq \bigcup_{t_k(t)\in\mathcal{D}_n} \left\{\sup_{0\leq t\leq \frac{h_k}{2}} \left|\sqrt{\kappa}B_t\right| > \frac{1}{4n+1}\right\}, \tag{3.44a}$$

and we see

$$\begin{aligned}\mathbb{P}\left(\epsilon > \frac{1}{4n+1}\right) &\leq \sum_{t_k(t)\in\mathcal{D}_n} \mathbb{P}\left(\sup_{0\leq t\leq \frac{h_k}{2}} \left|\sqrt{\kappa}B_t\right| > \frac{1}{4n+1}\right) \\ &\leq 2(4n+1)^3 \cdot e^{-(4n+1)/2\kappa}.\end{aligned} \tag{3.44b}$$

Conditioned on the event $\{\epsilon > (4n+1)^{-1}\}^c \in \Omega$, following *Eqn.* (3.41), we have

$$\sup_{t\in[0,1]} \left|Z_t(iy_n) - \widetilde{Z}_t^*(iy_n)\right| \leq \frac{1}{\sqrt{4n+1}}. \tag{3.45}$$

Hence, by the strict inclusion of events in probability space, we have our estimate to the perturbation term

$$\mathbb{P}\left(\sup_{t\in[0,1]} \left|Z_t(iy_n) - \widetilde{Z}_t^*(iy_n)\right| \leq \frac{1}{\sqrt{4n+1}}\right) \geq 1 - 2(4n+1)^3 \cdot e^{-(4n+1)/2\kappa}. \tag{3.46a}$$

We further observe that the splitting scheme $\widetilde{Z}_t(iy_n)$ coincides with the trajectory $\widetilde{Z}_t^*(iy_n)$ at the times $t_k(t) \in \mathcal{D}_n$, by virtue of *Lem.* 3.13. Hence we have the desired result

$$\mathbb{P}\left(E_{n,2}\right) \geq 1 - 2(4n+1)^3 \cdot e^{-(4n+1)/2\kappa}. \tag{3.46b}$$

□

Remark 3.15. *The perturbation in Prop.* 3.14 *in our splitting scheme is estimated via a probabilistic argument using ([15], Lem.* 2.2*). Notice that the time-reversed Loewner map* $h_t(z)$ *is different from the inverse of the forward map* $g_t^{-1}(z)$*, even though we do have the equality* $h_{T=1}(z) = g_{T=1}^{-1}(z)$*. In Sec.* 5*. we are going to briefly discuss the linear interpolation of driving force. To prove its convergence, we will need ([15], Lem.* 2.2*) again under a different context.*

Hence we have estimated the sup-norm on $[0, 1]$ of the second term in *Ineq.* (3.3*a*). Next, we will estimate the $\|\cdot\|_{[0,1],\infty}$ norm of the third term. Following *Eqn.* (2.10) with $L_0(z) = -2/z$ and $L_1(z) = \sqrt{\kappa}$, we could explicitly calculate, with $t_k(t) \le s < t_{k+1}$

$$\begin{aligned} \widetilde{Z}_s^{(0)} &= \exp\left(\frac{1}{2}(s - t_k(t))L_0\right)\widetilde{Z}_{t_k(t)} = \sqrt{\widetilde{Z}_{t_k(t)}^2 - 2(s - t_k(t))}, \\ \widetilde{Z}_s^{(1)} &= \exp\left(B_{t_k(t),s}L_1\right)\widetilde{Z}_{t_{k+1}}^{(0)} = \sqrt{\widetilde{Z}_{t_k(t)}^2 - 2h_k} + B_{t_k(t),s}, \\ \widetilde{Z}_s^{(2)} &= \exp\left(\frac{1}{2}(s - t_k(t))L_0\right)\widetilde{Z}_{t_{k+1}}^{(1)} = \sqrt{\left(\widetilde{Z}_{t_{k+1}}^{(1)}\right)^2 - 2(s - t_k(t))}, \end{aligned} \tag{3.47}$$

which could be written into the form via solving *Eqn.* (2.9)

$$\begin{aligned} &\widetilde{Z}_t(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n) \\ &= \frac{1}{2}\int_{t_k(t)}^t L_0(\widetilde{Z}_s^{(2)})ds + \int_{t_k(t)}^t L_1(\widetilde{Z}_s^{(1)})dB_s + \frac{1}{2}\int_{t_k(t)}^t L_0(\widetilde{Z}_s^{(0)})ds, \\ &= \int_{t_k}^t \sqrt{\kappa}dB_s - \int_{t_k}^t \frac{1}{\sqrt{(\widetilde{Z}_{t_k})^2 - 2(s - t_k)}}ds - \int_{t_k}^t \frac{1}{\sqrt{\left(\widetilde{Z}_{t_{k+1}}^{(1)}\right)^2 - 2(s - t_k)}}ds, \end{aligned} \tag{3.48a}$$

where t_k, t_{k+1} abbreviates $t_k(t), t_{k+1}(t)$ in the above expression, respectively. Solving these integrals, we have

$$\begin{aligned} &\widetilde{Z}_t(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n) \\ &= \sqrt{\kappa}B_{t_k(t),t} - \frac{2(t - t_k(t))}{\sqrt{\widetilde{Z}_{t_k(t)}^2 - 2(t - t_k(t))} + \widetilde{Z}_{t_k(t)}} \\ &- \frac{2(t - t_k)}{\sqrt{(\sqrt{\widetilde{Z}_{t_k}^2 - 2h_k} + \sqrt{\kappa}B_{t_k,t_{k+1}})^2 - 2(t - t_k)} + \sqrt{\widetilde{Z}_{t_k}^2 - 2h_k} + \sqrt{\kappa}B_{t_k,t_{k+1}}}. \end{aligned} \tag{3.48b}$$

Notice that *Eqn.* (3.49*b*) provides an exact form of the approximated process $\widetilde{Z}_t$, which leads us to the following result.

Proposition 3.16. *Let $\kappa \neq 8$. Let us consider the event*

$$E_{n,3} \coloneqq \left\{ \left\| \widetilde{Z}_t(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n) \right\|_{[0,1],\infty} \leq \frac{2}{n^{1/2}} + \frac{1}{n^{1/4}} \right\}. \tag{3.49}$$

Then as long as $||\mathcal{D}_n|| \leq n^{-1} \wedge (4n+1)^{-3}$, we have

$$\mathbb{P}\big(E_{n,3}\big) \geq 1 - \frac{1}{n} - 2e^{-\sqrt{n}/2\kappa}. \tag{3.50}$$

Proof. By ([3], *Sec.* 6.1), we have two general results $\operatorname{Im}(z) \leq \operatorname{Im}(\sqrt{z^2 - c})$ and $\operatorname{Im}(z) = \operatorname{Im}(z + c)$ for all $z \in \mathbb{H}$ and $c \in \mathbb{R}$. Applying this two results to *Eqn.* (3.49*b*), we have

$$\begin{aligned} \left| \widetilde{Z}_t(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n) \right| &\leq \left| \sqrt{\kappa} B_{t_k(t),t} \right| + \frac{h_k}{\operatorname{Im} \widetilde{Z}_{t_k(t)}(iy_n)} + \frac{h_k}{\operatorname{Im} \widetilde{Z}_{t_k(t)}(iy_n)}, \\ &\leq \left| \sqrt{\kappa} B_{t_k(t),t} \right| + \frac{2h_k}{y_n}, \end{aligned} \tag{3.51}$$

where the last inequality follows from ([1], *Lem.* 4.9), where it is shown that the map $t \mapsto \operatorname{Im} \widetilde{Z}_t(iy)$ is strictly increasing. Using ([9], *Cor.* 2.2), we obtain that

$$\mathbb{P}\left(\sup_{0 \leq t \leq h_k} \left| \sqrt{\kappa} B_t \right| \geq \frac{1}{n^{1/4}} \right) = 2\mathbb{P}\left(S_{h_k} \geq \frac{1}{\sqrt{\kappa} \cdot n^{1/4}} \right) \leq 2\sqrt{\frac{2}{\pi}} e^{-\frac{n^{-1/2}}{2h_k \cdot \kappa}}, \tag{3.52}$$

by reflection principle. Note that we have restricted $h_k \leq n^{-1} \wedge (4n+1)^{-3}$ for all $t_k(t) \in \mathcal{D}_n$. In this regard

$$\mathbb{P}\left(\sup_{t \in [0,1]} \left| \sqrt{\kappa} B_{t_k(t),t} \right| \geq \frac{1}{n^{1/4}} \right) \leq 2e^{-\sqrt{n}/2\kappa}, \tag{3.53}$$

and

$$\mathbb{P}\big(E_{n,3}\big) \geq 1 - \frac{1}{n} - 2e^{-\sqrt{n}/2\kappa}. \tag{3.54}$$

□

At this point, we have evaluated the $\|\cdot\|_{[0,1],\infty}$ norm *w.r.t.* all the three terms in *Ineq.* (3.3*a*). Therefore, we come to prove the main result.

Proof. (*of Thm. 3.4*) Denote $E_{n,4} \coloneqq E^{**}_{n,1}$. On the event $E_{n,1} \cap E_{n,2} \cap E_{n,3} \cap E_{n,4} \subset \Omega$, we observe from *Prop.* 3.10, *Prop.* 3.11, *Prop.* 3.14, and *Prop.* 3.16

that

$$
\begin{aligned}
&\|\eta(t)-Z_t(iy_n)\|_{[0,1],\infty} \le \frac{1}{n^{(1-\epsilon_0)/4}},\\
&\left\|Z_t(iy_n)-Z_{t_k(t)}(iy_n)\right\|_{[0,1],\infty} \le \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}}+\frac{2}{n^{(1-\epsilon_0)/4}},\\
&\left\|Z_{t_k(t)}(iy_n)-\widetilde{Z}_{t_k(t)}(iy_n)\right\|_{[0,1],\infty} \le \frac{1}{(4n+1)^{1/2}},\\
&\left\|\widetilde{Z}_{t_k(t)}(iy_n)-\widetilde{Z}_t(iy_n)\right\|_{[0,1],\infty} \le \frac{2}{n^{1/2}}+\frac{1}{n^{1/4}},
\end{aligned}
\tag{3.55}
$$

given $||\mathcal{D}_n|| \le n^{-1} \wedge (4n+1)^{-3}$ and $\beta_1 \in (0,1)$. If we define

$$
\begin{aligned}
\varphi_1(n) &:= \frac{2\phi(\sqrt{n})}{(1-\beta_1)n^{(1-\beta_1)/2}}+\frac{3}{n^{(1-\epsilon_0)/4}}+\frac{1}{(4n+1)^{1/2}}+\frac{2}{n^{1/2}}+\frac{1}{n^{1/4}} \to 0,\\
\varphi_2(n) &:= \frac{1}{n}+\frac{c_2}{n^2}+\frac{c_4}{n^{c_3/2}}+2(4n+1)^3\cdot e^{-(4n+1)/2\kappa}+2e^{-\sqrt{n}/2\kappa}+3\epsilon_n \to 0,
\end{aligned}
\tag{3.56}
$$

as $n \to \infty$. Then the following inequality proves our result

$$
\mathbb{P}\left(\left\|\eta(t)-\widetilde{Z}_t(iy_n)\right\|_{[0,1],\infty} \le \varphi_1(n)\right) \ge \mathbb{P}\left(E_{n,1}\cap\cdots\cap E_{n,4}\right) \ge 1-\varphi_2(n). \tag{3.57}
$$

□

Corollary 3.17. *For almost all $\omega \in \Omega$, we have that*

$$
\left\|\eta(t)-\widetilde{Z}_t(iy_n)\right\|_{[0,1],\infty} \to 0, \text{ with } n \to \infty. \tag{3.58}
$$

Proof. This corollary immediately follows from the strong convergence in probability in $Thm.$ 3.4. □

Remark 3.18. *From a practical view-point, for a fixed level $n \in \mathbb{N}$, in order to achieve better precision one can choose a constant $\tau > 0$, called tolerance, to ensure that*

$$
\left|\widetilde{Z}_{t_{k+1}}-\widetilde{Z}_{t_k}\right| \le \tau \tag{3.59}
$$

for each k. To achieve this, we start by computing $\widetilde{Z}_t$ along a prior uniform partition until $\left|\widetilde{Z}_{t_{k+1}}-\widetilde{Z}_{t_k}\right| > \tau$. If this event occurs, we reduce the step size h_k of the SLE_κ discretization, that is we insert the mid-point of this interval $[t_k(t), t_{k+1}]$ into the partition.

Notice that the choice of a refined partition actually depends on $\omega \in \Omega$ because the evolution $(\widetilde{Z}_t)_{0\le t\le 1}$ contains Brownian motion.

4 Convergence in the L^p norm under assumptions

Following *Ineq.* (3.3a) and *Ineq.* (3.3b), we are going to estimate the L^p norm to $\big(\eta(t) - \widetilde{Z}_t(iy_n)\big)$ with $p \geq 2$, given some technical assumptions. This approach can be thought of as a variant of the convergence, under the mentioned assumptions. We plan to investigate the proof of these assumptions in future work. Similar to the proof of strong convergence in probability, the first step is to estimate each of the four terms in *Eqn.* (3.56) individually.

Assumption 4.1. *There exists $p_1 \geq 2$ and $\epsilon_0 \in (0,1)$ such that*

$$\sup_{t\in[0,1]} |\eta(t) - Z_t(iy_n)| \leq C_1(\omega) \cdot y_n^{1-\epsilon_0}, \tag{4.1}$$

where the constant $C_1(\omega)$ is almost surely finite as in ([3], Lem. 6.7). Moreover, we assume that $C_1(\omega)$ is p_1-integrable, i.e. $C_1(\omega) \in L^{p_1}(\mathbb{P})$.

Assumption 4.2. *There exists $p_2 \geq 2$ and $\beta_2 \in (0,1)$ such that the SLE_κ Loewner chain is generated by a curve when $\kappa \neq 8$ with the following modulus of continuity*

$$|\eta(t+s) - \eta(t)| \leq C_2(\omega) s^{(1-\beta_2)/2} \tag{4.2}$$

where the constant $C_2(\omega)$ is almost surely finite as in ([5], Prop. 4.3). And moreover $C_2(\omega)$ is p_2-integrable, i.e. $C_2(\omega) \in L^{p_2}(\mathbb{P})$.

Assumption 4.3. *The Ninomiya-Victoir Splitting Scheme satisfies the various regularity assumptions, including the ellipticity condition in* ([7], *Rmk.* 4.1) *and then admits the inequality*

$$\mathbb{E}\left[\sup_{t\in[0,1]} \left|Z_t(iy_n) - \widetilde{Z}_t(iy_n)\right|^p\right] \leq \frac{c_6}{n^p}, \tag{4.3}$$

for some constant $c_6 > 0$, and for all $p \geq 2$.

If these assumptions hold as expected, we could take a crucial step near the idea of L^p convergence of our splitting scheme. In this section, we choose $p \coloneqq p_1 \wedge p_2 \geq 2$. We have the following propositions.

Proposition 4.4. *Admitting Asmp. 4.1 with $||\mathcal{D}_n| \leq n^{-1} \wedge (4n+1)^{-3}$ where $||\mathcal{D}_n||$ is refined partition, then there exists a decreasing function $\psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi_1(n) \to 0$ as $n \to \infty$, and*

$$\mathbb{E}\left[\int_0^1 |\eta(t) - Z_t(iy_n)|^p dt\right] \leq \psi_1(n). \tag{4.4}$$

Proof. By ([3], *Lem.* 6.7), there exists $\epsilon_0 \in (0,1)$ such that *almost surely*

$$\sup_{t\in[0,1]} |\eta(t) - Z_t(iy_n)| \le C_1(\omega)\cdot y_n^{1-\epsilon_0}. \tag{4.5}$$

It is clear then

$$\begin{aligned}\mathbb{E}\Big[\int_0^1 |\eta(t) - Z_t(iy_n)|^p dt\Big] &\le \mathbb{E}\Big[\|\eta(t) - Z_t(iy_n)\|^p_{[0,1],\infty}\Big] \\ &\le \mathbb{E}\big[C_1(\omega)^p\big]\cdot \frac{1}{n^{(1-\epsilon_0)p/2}} := \psi_1(n) \overset{n\to\infty}{\longrightarrow} 0.\end{aligned} \tag{4.6}$$

□

Proposition 4.5. *Admitting Asmp.* 4.2 *with* $||\mathcal{D}_n|| \le n^{-1}\wedge(4n+1)^{-3}$*, there exists a decreasing function* $\psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\psi_2(n)\to 0$ *as* $n\to\infty$*, and*

$$\mathbb{E}\Big[\int_0^1 \big|Z_t(iy_n) - Z_{t_k(t)}(iy_n)\big|^p dt\Big] \le \psi_2(n). \tag{4.7}$$

Proof. ([5], *Prop.* 3.8) and ([5], *Prop.* 4.3) imply that there exists $\beta_2 \in (0,1)$ such that

$$\sup_{t\in[0,1]} \big|\eta(t) - \eta\big(t_k(t)\big)\big| \le C_2(\omega)\cdot\frac{1}{n^{(1-\beta_2)/2}}. \tag{4.8}$$

By *Prop.* 4.4, it follows that

$$\sup_{t\in[0,1]} \big|Z_t(iy_n) - Z_{t_k(t)}(iy_n)\big| \le \frac{2C_1(\omega)}{n^{(1-\epsilon_0)/2}} + \frac{C_2(\omega)}{n^{(1-\beta_2)/2}}. \tag{4.9}$$

It is clear then

$$\begin{aligned}\mathbb{E}\Big[\int_0^1 \big|Z_t(iy_n) - Z_{t_k(t)}(iy_n)\big|^p dt\Big] &\le \mathbb{E}\Big[\big\|Z_t(iy_n) - Z_{t_k(t)}(iy_n)\big\|^p_{[0,1],\infty}\Big] \\ &\le \mathbb{E}\big[C_1(\omega)^p\big]\cdot\frac{2^{2p-1}}{n^{(1-\epsilon_0)p/2}} + \mathbb{E}\big[C_2(\omega)^p\big]\cdot\frac{2^{p-1}}{n^{(1-\beta_2)p/2}} := \psi_2(n) \overset{n\to\infty}{\longrightarrow} 0.\end{aligned} \tag{4.10}$$

□

Proposition 4.6. *Admitting Asmp.* 4.3 *with* $||\mathcal{D}_n|| \le n^{-1}\wedge(4n+1)^{-3}$*, it is immediate that*

$$\mathbb{E}\Big[\sup_{t\in[0,1]} \Big|Z_{t_k(t)}(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n)\Big|^p\Big] \le \frac{c_6}{n^p}. \tag{4.11}$$

If we denote $\psi_3(n) \coloneqq c_6/n^p$, *then*

$$\mathbb{E}\bigg[\int_0^1 \Big|Z_{t_k(t)}(iy_n) - \widetilde{Z}_{t_k(t)}(iy_n)\Big|^p dt\bigg] \leq \psi_3(n) \overset{n\to\infty}{\longrightarrow} 0. \tag{4.12}$$

Proof. The proposition follows from *Eqn.* (4.9). □

To give an estimate to the last term in *Eqn.* (3.56), we quote the known interpolation inequality from ([8], *Sec.* 6.5) for Lebesgue spaces and another inequality *w.r.t.* supremum Brownian motion.

Lemma 4.7. *For all* $1 < p < r < q$, *suppose* $f \in L^p \cap L^q$. *Then* $f \in L^r$ *with*

$$\|f\|_r \leq \big(\|f\|_p\big)^{(1/r-1/q)/(1/p-1/q)} \big(\|f\|_q\big)^{(1/r-1/q)/(1/p-1/q)}. \tag{4.13}$$

Lemma 4.8. *For all* $m \in \mathbb{N}$, *we have*

$$\mathbb{E}\bigg[\sup_{s\in[0,t]} |B_s|^{2m}\bigg] = \pi^{-\frac{1}{2}}\Gamma\bigg(\frac{1}{2}+m\bigg)2^m \cdot t^m, \tag{4.14}$$

where B_t *is a standard one-dimensional Brownian motion.*

Proof. The proof follows if we revisit the supremum Brownian motion $S_t = \sup_{0\leq s\leq t} B_s$ from ([9], *Cor.* 2.2), by computing all even order moments. □

Proposition 4.9. *With* $||\mathcal{D}_n|| \leq n^{-1} \wedge (4n+1)^{-3}$, *there exists a decreasing function* $\psi_4 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\psi_4(n) \to 0$ *with* $n \to \infty$, *and*

$$\mathbb{E}\bigg[\int_0^1 \Big|\widetilde{Z}_{t_k(t)}(iy_n) - \widetilde{Z}_t(iy_n)\Big|^p dt\bigg] \leq \psi_4(n). \tag{4.15}$$

Proof. By *Prop.* 3.14 and *Eqn.* (3.52), we know *almost surely* that

$$\Big|\widetilde{Z}_{t_k(t)}(iy_n) - \widetilde{Z}_t(iy_n)\Big|^p \leq 2\big|\sqrt{\kappa}B_{t_k(t),t}\big|^p + \frac{2^{p+1}}{n^{p/2}}. \tag{4.16}$$

Since $p \geq 2$, there exists $\{m, m+1\} \subset \mathbb{N}$ so that $2m \leq p < 2(m+1)$. Remember that $t - t_k(t) \leq h_k \leq n^{-1}$. Then, by *Lem.* 4.8

$$\begin{aligned} \mathbb{E}\bigg[\sup_{t\in[0,1]} \big|\sqrt{\kappa}B_{t_k(t),t}\big|^{2m}\bigg] &\leq \frac{2^m\kappa^m\Gamma\big(\frac{1}{2}+m\big)}{\pi^{\frac{1}{2}} \cdot n^m}, \\ \mathbb{E}\bigg[\sup_{t\in[0,1]} \big|\sqrt{\kappa}B_{t_k(t),t}\big|^{2m+2}\bigg] &\leq \frac{2^{m+1}\kappa^{m+1}\Gamma\big(\frac{3}{2}+m\big)}{\pi^{\frac{1}{2}} \cdot n^{m+1}}. \end{aligned} \tag{4.17}$$

By *Lem.* 4.7, we use the interpolation inequality in Lebesgue spaces and have

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,1]}\big|\sqrt{\kappa}B_{t_k(t),t}\big|^p\Big]\\
&\le \mathbb{E}\Big[\sup_{t\in[0,1]}\big|\sqrt{\kappa}B_{t_k(t),t}\big|^{2m}\Big]^{m+1-\frac{p}{2}}\mathbb{E}\Big[\sup_{t\in[0,1]}\big|\sqrt{\kappa}B_{t_k(t),t}\big|^{2m+2}\Big]^{(m+1-\frac{p}{2})\cdot\frac{m}{m+1}}\\
&\le \frac{c_7}{n^{m(m+1-\frac{p}{2})}} := \psi_4(n)\overset{n\to\infty}{\longrightarrow} 0,
\end{aligned}
\tag{4.18}
$$

where $c_7 > 0$ is a constant depending only on $p \ge 2$. □

We have, at this point, estimated the L^p norm *w.r.t.* the four terms in *Eqn.* (3.52) under several assumptions. Combining them together, under the specified assumptions, we obtain the following result.

Theorem 4.10. *Admitting Asmp.* 4.1, 4.2, *and* 4.3*, together with* $||\mathcal{D}_n|| \le n^{-1}\wedge(4n+1)^{-3}$*, there exists a decreasing function* $\psi:\mathbb{R}_+\to\mathbb{R}_+$ *such that* $\psi(n)\to 0$ *as* $n\to\infty$*, and*

$$\mathbb{E}\Big[\int_0^1\Big|\eta(t)-\widetilde{Z}_t(iy_n)\Big|^p\Big]\le\psi(n). \tag{4.19}$$

Proof. Define $\psi := \psi_1+\psi_2+\psi_3+\psi_4$, then the result follows. □

5 Linear interpolation of the driving force

In this last section we study the approximation obtained from the piece-wise linear interpolation of the Brownian driver. This method is independent from the study of the splitting scheme in the first part of the paper. Let us briefly discuss that idea to simulate the driving force $\lambda(t)$ and its corresponding hull K_t, for all $t\in[0,T]$. The interpolation algorithm is based on the following observations. Fix $s>0$, let $(\widetilde{g}_t)_{0\le t\le T}$ be the Loewner chain driven by the continuous driving force $\widetilde{\lambda}(t)=\lambda(s+t)$, with $0\le t\le T-s$. Using the format of the forward Loewner Differential equation, we have that

$$\partial_t g_{s+t}\circ g_s^{-1}(z)=\frac{2}{g_{s+t}\circ g_s^{-1}(z)-\lambda(s+t)}=\frac{2}{g_{s+t}\circ g_s^{-1}(z)-\widetilde{\lambda}(t)}, \tag{5.1}$$

and $g_s\circ g_s^{-1}(z)=z$ with $z\in\mathbb{H}\backslash K_s$. The uniqueness property of Loewner chains implies that $\widetilde{g}_t(z)=g_{s+t}\circ g_s^{-1}(z)$. We denote $\widetilde{K}_t$ to be the hull associated with the Loewner chain $\widetilde{g}_t$, indeed

$$\mathbb{H}\cap g_s(K_{s+t})=\widetilde{K}_t \text{ and } K_{s+t}=K_s\cup g_s^{-1}(\widetilde{K}_t). \tag{5.2}$$

Hence, computing K_s, g_s^{-1} and $\widetilde{K}_t$ would enable us to compute $\widetilde{K}_{s+t}$.

In order to implement the linear-interpolation of the Brownian driver, we consider the following driver that linearly interpolates $\lambda_t = B_t$.

$$\lambda^n_{linear}(t) \coloneqq n\big(\lambda(t_{k+1}) - \lambda(t_k)\big)(t - t_k) + \lambda(t_k) \text{ on } [t_k, t_{k+1}], \tag{5.3}$$

In [11] there was previously considered the case of square-root interpolation of the Brownian driver

$$\lambda^n_{square\text{-}root}(t) \coloneqq \sqrt{n}\big(\lambda(t_{k+1}) - \lambda(t_k)\big)\sqrt{t - t_k} + \lambda(t_k) \text{ on } [t_k, t_{k+1}], \tag{5.4}$$

in order to approximate the SLE traces. These methods are based on the fact that for driers of the form $c\sqrt{t} + d$ for $c, d, \in \mathbb{R}$ and ct for $c \in \mathbb{R}$ the Loewner maps and curves can be explicitly computed (see [10]). We refer the reader to [11] for more details.

Definition 5.1. *As before, $\mathcal{D}_n \coloneqq \{t_0 = 0, t_1, \ldots, t_n = 1\}$ will denote a uniform partition on the time interval $[0, 1]$.*

Definition 5.2. *Given the Brownian sample paths $B_{\cdot}(\omega) : [0, 1] \to \mathbb{R}$, we write our linear interpolation in the following form*

$$\lambda^n(t) \coloneqq n\big(\sqrt{\kappa}B_{t_{k+1}} - \sqrt{\kappa}B_{t_k(t)}\big)(t - t_k(t)) + \sqrt{\kappa}B_{t_k(t)} \text{ on } [t_k(t), t_{k+1}]. \tag{5.5}$$

Definition 5.3. *Following the convention in Sec. 2. and Sec. 3. we use the notation $\gamma : [0, 1] \to \mathbb{H} \cup \{\sqrt{\kappa}B_1\}$ for the forward Loewner curve generated by the forward Loewner chain $(g_t)_{t\in[0,1]}$ with driving force $\sqrt{\kappa}B_t$. We further let $\gamma^n : [0, 1] \to \mathbb{H} \cup \{\sqrt{\kappa}B_1\}$ to denote the forward Loewner curve generated by the forward Loewner chain $(g^n_t)_{t\in[0,1]}$, driven by the piecewise-linear force $\lambda^n(t)$.*

Remark 5.4. *Notice that in this section we will interpolate the forward Loewner chain and hence simulate the forward Loewner curve $\gamma(t)$, whereas we have simulated the backward Loewner curve $\eta(t)$ via Ninomiya-Victoir Scheme in Sec. 2.*

Definition 5.5. *With $(g^n_t)_{t\in[0,1]}$ the Loewner chain corresponding to $\lambda^n(t)$, let $f^n_t : \mathbb{H} \to \mathbb{H}\backslash\gamma^n([0, t])$ be the inverse map of $g^n_t(z)$ and denote $\widehat{f}^n_t(z) \coloneqq f^n_t\big(z + \lambda^n(t)\big)$. Choose $G^n_k \coloneqq (\widehat{f}^n_{t_k(t)})^{-1} \circ \widehat{f}^n_{t_{k+1}}$. Then*

$$\widehat{f}^n_{t_k(t)} = G^n_0 \circ G^n_1 \circ \cdots \circ G^n_{k-1}. \tag{5.6}$$

Definition 5.6. *Choose $\gamma^n_t(s) \coloneqq g^n_t\big(\gamma^n(t + s)\big)$ with $s \in [0, 1 - t]$, for all*

$t \in [0,1]$.

Lemma 5.7. *Consider the event* $F_{n,1} := E'_{n,1}$ *and* $F_{n,2} := E''_{n,1}$ *as in Eqn.* (3.4) *and in Eqn.* (3.6). *Then we have*

$$\mathbb{P}\big(F_{n,1}\big) \geq 1 - \frac{c_2}{n^2} \quad \textit{and} \quad \mathbb{P}\big(F_{n,2}\big) \geq 1 - \frac{c_4}{n^{c_3/2}}, \tag{5.7}$$

where $c_2, c_3, c_4 > 0$ *are constants depending only on* $\kappa \neq 8$.

Theorem 5.8. *Let* γ *be the* SLE_κ *trace and let* γ^n *be the trace obtained by the linear interpolation of the Brownian driver. There exist* $c_6, c_7 > 0$ *depending only on* $\kappa \neq 8$ *such that if we consider the event*

$$F_n := \left\{ \|\gamma - \gamma^n\|_{[0,1],\infty} \leq \frac{c_6 (\log n)^{c_7}}{n^{(1-\sqrt{(1+\beta_1)/2})/2}} \right\}. \tag{5.8}$$

Then we have $\mathbb{P}\big(F_n\big) \geq 1 - c_2 \cdot n^{-2} - c_3 \cdot n^{-c_4/2}$.

This theorem is our main result in the context of linearly interpolating Brownian driver. We will not give a detailed proof here because the proof is similar to the known result of square-root interpolation obtained in ([11], *Sec.* 2.). Instead, we outline the ideas to estimate the convergence in probability of the linear interpolation method.

On the event $F_{n,1} \cap F_{n,1}$, we want to give a uniform bound to $|\gamma(t) - \gamma^n(t)|$ with $t \in [0,1]$. In fact, for all $t_k(t) \in \mathcal{D}_n$, we write

$$\begin{aligned} |\gamma(r + t_k(t)) - \gamma^n(r + t_k(t))| &\leq |\gamma(r + t_k(t)) - \gamma(s + t_k(t))| + \left|\widehat{f}_{t_k(t)}(z) - \widehat{f}^n_{t_k(t)}(w)\right| \\ &\leq |\gamma(r + t_k(t)) - \gamma(s + t_k(t))| + \left|\widehat{f}_{t_k(t)}(z) - \widehat{f}_{t_k(t)}(w)\right| + \left|\widehat{f}_{t_k(t)}(w) - \widehat{f}^n_{t_k(t)}(w)\right|, \end{aligned} \tag{5.9}$$

where $w := \gamma^n_k(r)$, r is arbitrarily fixed in $[\frac{1}{n}, \frac{2}{n}]$ and $z := \gamma_k(s)$ is chosen to be the highest point in the arc $\gamma_k([0, \frac{2}{n}])$. The first term in *Eqn.* (5.5) is bounded by the uniform continuity of $\gamma(t)$ on the event $F_{n,1}$.

The estimate of the second term in *Eqn.* (5.5) is comparing the images of nearby points in $\mathbb{H}$ very close to the real and the imaginary axis, under a conformal map. To proceed our discussion, we introduce, for any subpower function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, constant $c > 0$, and integer $n \in \mathbb{N}_+$ that

$$A_{n,c,\phi} := \left\{ x + iy \in \mathbb{H};\ |x| \leq \frac{\phi(n)}{\sqrt{n}} \text{ and } \frac{1}{\sqrt{n}\phi(n)} \leq y \leq \frac{c}{\sqrt{n}} \right\}, \tag{5.10}$$

which is a box near the origin in the upper half-plane. The reason we introduce this extra object is that the images of nearby points in $A_{n,c,\phi}$ will also be close to each other under certain conformal maps, in the sense of the following lemmas.

Lemma 5.9. ([11], *Lem.* 2.6) *There exist constants $\alpha > 0$, and $c' > 0$, depending only on $c > 0$ in the definition of the box $A_{n,c,\phi}$, such that for all $z_1, z_2 \in A_{n,c,\phi}$ and conformal map $f : \mathbb{H} \to \mathbb{C}$, we have*

$$\begin{aligned} |f'(z_1)| &\leq c'\phi(n)^{\alpha} \cdot |f'(i \operatorname{Im} z_1)|, \\ d_{\mathbb{H},hyp}(z_1, z_2) &\leq c' \log \phi(n) + c', \end{aligned} \tag{5.11}$$

where $d_{\mathbb{H},hyp}(z_1, z_2)$ denotes the hyperbolic distance between z_1 and z_2 in $\mathbb{H}$.

Lemma 5.10. ([16], *Cor.* 1.5) *Suppose $f : \mathbb{H} \to \mathbb{C}$ is a conformal map, then for all $z_1, z_2 \in \mathbb{H}$, we have*

$$|f(z_1) - f(z_2)| \leq 2|(\operatorname{Im} z_1) f'(z_1)| \cdot \exp\big(4 d_{\mathbb{H},hyp}(z_1, z_2)\big). \tag{5.12}$$

Therefore, it is natural that we want to show $\{z, w\} \in A_{n,c,\phi}$ with proper parameters. Indeed, this is the case in the square-root interpolation ([11], *Lem.* 3.3). The only non-trivial remark is the following.

Remark 5.11. *In the linear interpolation, from ([10], Sec. 3.) we know that for a typical linear force $\lambda(t) = t$ on the time interval $[0, \infty)$, the Loewner curve admits the form*

$$t \mapsto 2 - 2\rho_t \cot \rho_t + 2i\rho_t, \tag{5.13}$$

where ρ_t increases monotonously from $\rho_0 = 0$ to $\rho_\infty = \pi$. Indeed, the Loewner curve of a general linear force $t \mapsto at + b$ requires some change of constant parameters depending on $a, b \in \mathbb{R}$, possibly with change in signs. Hence, we know the arc $\gamma_k^n : [0, \frac{1}{n}] \to \mathbb{H} \cup \{0\}$ corresponding to the piecewise-linear force $\lambda^n(t_k(t) + t) - \lambda^n(t_k(t))$ with $t \in [0, \frac{1}{n}]$ has an image which vertically stretches monotonically upward and horizontally either leftward or rightward. Hence, the images $\gamma_k^n([0, \frac{1}{n}])$ attains its maximal height at its tip $\gamma_k^n(\frac{1}{n})$, which justifies our choice of $z = \gamma_k(s)$.

In this regard we could use *Lem.* 5.9 and *Lem.* 5.10 to give an upper bound to the second term in *Ineq.* (5.5).

Now, let us turn our attention to the third term in *Ineq.* (5.5). This is actually a perturbation term: we need to measure the difference of one point in $\mathbb{H}$ under two conformal maps. In order to estimate the third term in *Ineq.* (5.5), we use the following result.

Lemma 5.12. ([15], *Lem.* 2.2) *Let $0 < T < \infty$. Suppose $f_t^{(1)}$ and $f_t^{(2)}$ are the inverse map to the forward Loewner chain satisfying Eqn. (2.1) with driving force $W_t^{(1)}$ and $W_t^{(2)}$, respectively. Define $\epsilon \coloneqq \sup_{s \in [0,T]} \left| W_s^{(1)} - W_s^{(2)} \right|$. Then if*

$u = x + iy \in \mathbb{H}$, *we have*

$$\left|f_T^{(1)} - f_T^{(2)}\right| \leq \epsilon \exp\left\{\frac{1}{2}\left[\log\frac{I_{T,y}\left|\partial_z f_T^{(1)}(u)\right|}{y}\log\frac{I_{T,y}\left|\partial_z f_T^{(2)}(u)\right|}{y}\right]^{\frac{1}{2}} + \log\log\frac{I_{T,y}}{y}\right\}, \tag{5.14}$$

where $I_{T,y} \coloneqq \sqrt{4T + y^2}$.

Applying *Lem.* 5.12 to estimate the third term in *Ineq.* (5.5), combining the estimate to the first and the second terms, on the event $F_{n,1} \cap F_{n,2}$, we see that

$$\sup_{t\in[0,1]} |\gamma(t) - \gamma^n(t)| \leq \frac{c_6(\log n)^{c_7}}{n^{(1-\sqrt{(1+\beta_1)/2})/2}}, \tag{5.15}$$

for $\beta_1 \in (0, 1)$. By definition of the event $F_n \in \Omega$, we know $F_{n,1} \cap F_{n,2} \subset F_n$. Moreover using the probability bounds on the events $F_{n,1}$ and $F_{n,2}$ in *Ineq.* (5.3), we obtain *Thm.* 5.8.

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