Introduction to Rough Paths Theory with applications to the expected signature of SLE_κ for $\kappa \in [0, 4]$ (as developed by Brent Werness)

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Introduction

The Loewner equation was introduced by Charles Loewner in 1923 in Complex Analysis and Geometric Function Theory and it played an important role in the proof of the Bieberbach Conjecture by Louis de Branges in 1985 . There are two versions of Loewner equation -radial and chordal- and they can be written as a partial differential equations or ordinary differential equations depending on the family of conformal maps that are studied. Moreover, the ordinary differential equation versions can be studied by considering a reverse time evolution also. In 2000 , Oded Schramm introduced a stochastic version of the Loewner equation in order to study the scaling limits of planar loop erased random walk and uniform spanning tree.

One motivation for studying the processes SLE_κ is their success in describing the scaling limits of various discrete models from planar Statistical Physics. For instance, it was proved that the scaling limit of loop erased random walk (with the loops erased in a chronological order) converges in the scaling limit to SLE_κ with $\kappa = 2$. Moreover, other two dimensional discrete models from Statistical Mechanics including Ising model cluster boundaries, Gaussian free field interfaces, percolation on the triangular lattice at critical probability, and Uniform spanning trees converge in the scaling limit to SLE_κ for values of $\kappa = 3$, $\kappa = 4$, $\kappa = 6$ and $\kappa = 8$ respectively. In fact, the use of Loewner equation along with the techniques of stochastic calculus, in this context gave a precise meaning to the passage to the scaling limit itself and proved rigorously the conformal invariance of the limits.
In addition, the introduction of the SLE gave new insights about some old problems. For instance, there were proved some long-standing open problems about planar Brownian motion such as the Mandelbrot Conjecture about the Hausdorff dimension of the Brownian frontier.

The Theory of Rough Paths was introduced by Lyons in [4]. The Theory provides a deterministic platform to study Stochastic Differential Equations which extends both Young's and stochastic integration theory beyond regular functions and semi-martingales . In short, Rough Paths Theory provides a method of constructing solutions to differential equations driven by paths that are not of bounded variation but have controlled roughness. We introduce step by step the necessary ingredients and terminology to characterize the roughness of a path and to give precise meaning to natural objects that appear in the study of paths that are not smooth. The final goal is to give precise meaning to the notion of solution of a differential equation with rough driver.

Introduction to Rough Paths Theory

In this section we introduce the basic notions in the Theory of Rough Paths such as p-variation, Young's Theory of integration, the notion of signature of a path and the underlying tensor algebra, the definition of a Rough Path and concepts related with Rough Differential Equations. The main references for the material presented in this Chapter are Lyons, Lévy and Caruana (insert citation) along with Friz and Victoir (insert citation).

The signature of a path is a way of summarize the information needed to solve differential equations driven by paths. In the one dimensional case, in order to solve a linear differential equation of the form $dY_t = f(Y_t)dX_t$ all we need to know about the one-dimensional paths X_t is its end-point. Furthermore, in higher dimensions more information is needed and this is captured in the sequence of iterated integrals of the path. The collection of this iterated integrals of the path is the *Signature of the Path.*

The Riemann-Stieltjes integral gives meaning to the object $\int XdY$, where $X, Y : [0,T] \to \mathbb{R}$ are continuous functions (paths) with Y having bounded variation. Moreover, due to Young this result may be extended to a much rich class of regular functions.
Let $X_{[s,t]}$ denote the restriction of the path X to the compact interval $[s,t]$. We introduce the notion of p-variation.

Definition 0.2.1. *Let (E,d) be a metric space. The p-variation of a path $X : [0,T] \to E$ is defined by*

$$||X_{[0,T]}||_{p-var} := \sup_{\mathcal{D}=(t_0,t_1,\ldots,t_n)\subset[0,T]} \left(\sum_{i=0}^{n-1} d(X_{t_i}, X_{t_{i+1}})^p \right)^{\frac{1}{p}} ,$$

where the supremum is taken over all finite partitions of the interval $[0,T]$.

Unless stated, the metric space E is a finite dimensional real vector space V with dimension d and basis vectors $e_1, \ldots, e_d$.
Throughout the Thesis, we use the notation $X_{s,t} = X_t - X_s$.

The notion of control is used in order to understand the p-variation of paths. Also, this notion is of great importance in the definition of rough paths.
Let $\Delta_T = \{(s,t) | 0 \leqslant s \leqslant t \leqslant T\}$.

Definition 0.2.2. *A control on $[0,T]$ is a non-negative continuous function $\omega : \Delta_T \to [0,\infty)$ for which*

$$\omega(s,t) + \omega(t,u) \leqslant \omega(s,u),$$

for all $0 \leqslant s \leqslant t \leqslant u \leqslant T$, and $\omega(t,t) = 0$, for all $t \in [0,T]$.

For a path $X : [0,T] \to \mathbb{E}$ of finite p-variation, define $\omega_X(s,t) := ||X_{[s,t]}||^p_{p-var}$. To check that this quantity indeed defines a control it requires to check both proprieties along with the continuity (the continuity is proved in Section 1.2.2 of (insert citation) and Proposition 5.8 of (insert citation)). Moreover, this specific control is used to provide a reparametrization of the interval $[0,T]$ such that X becomes Hölder continuous with exponent $\frac{1}{p}$. To see this, consider the function $\tau : [0,T] \to [0,T]$ to be the inverse of the function $t \to \omega_X(0,t)\frac{T}{\omega_X(0,T)}$. Using this inverse function, we obtain that $d(X_{\tau(s)}, X_{\tau(t)})^p \leqslant \frac{\omega_X(0,T)}{T}(t-s)$.

0.2.1. Young integration. In what follows next, let us consider E to be a Banach space and X, Y to be continuous functions (paths) with values in a Banach spaces. A first extension to the usual Riemann-Stieltjes integral is given by Young's Theory of Integration. Using the results of Young, we can make sense of $\int_0^t Y_s dX_s$ provided that X and Y have finite p-variation and finite q-variation respectively with $\frac{1}{p} + \frac{1}{q} > 1$. Note that the bounded variation (i.e. $p = 1$) of X is contained in Young's theory of integration. To be precise, we have the result

Theorem 0.2.3. *Let V and W be Banach spaces and $X : [0, T] \to V$ and $Y : [0, T] \to \mathbf{L}(V, W)$ be two paths of finite p-variation and q-variation respectively with $\frac{1}{p} + \frac{1}{q} > 1$. Then, the limit*

$$\lim_{|\mathcal{D}| \to 0, \mathcal{D} \subset [0,t]} \sum_{i=0}^{n-1} Y_{t_i}(X_{t_i, t_{i+1}})$$

exists for all $t \in [0, T]$ and we define $\int_0^t Y_s dX_s$ as this limit. Furthermore, as a path in W , $\int_0^{\cdot} Y_s dX_s$ has finite p-variation.

Note that the Theorem holds independent of the sequence of partitions chosen provided that the size of the mesh $|\mathcal{D}| = \max_{0 \leqslant i \leqslant n-1} t_{i+1} - t_i \to 0$.

Differential equations driven by signals with finite p-variation for $p < 2$

In the classical theory of Ordinary Differential Equations, Cauchy-Peano Theorem asserts that when X has bounded variation, if the vector field f is a continuous function, then the differential equation $dY_t = f(Y_t)dX_t$ has solutions. However, this result does not in general guarantee uniqueness. In order to have uniqueness of solution, the Theorem of Picard-Lindelöf tells us that we have to impose more conditions on the function f, i.e. we need f to be Lipschitz continuous. In order to extend this results for signals with finite p-variation for $1 \leqslant p < 2$. we need to presumably have to use a smoother class of vector fields f than the continuous ones. For example, in order to make sense of the integral $\int f(Y)dX$, we need that $f(Y)$ has finite q-variation for some q such that $\frac{1}{p} + \frac{1}{q} > 1$. It is clear that not every continuous function does that, but we can restrict to the specific class of functions with this property.
The main result of this Section, presented in (insert reference Lyons Caruana) is a version of the classical Picard Fix Point Theorem in the context of differential equations driven by signals with finite p-variation for some $p < 2$. In order to state it, we need to introduce the notion of $Lip(\gamma)$ function. For this we consider again the metric space E to be a Banach space.

Definition 0.3.1. *Let V and W be two Banach spaces. Let $k \geqslant 0$ be an integer. Let $\gamma \in (k, k+1]$ be a real number. Let F be a closed subset of V . Let $f : F \to W$ be a function. For each integer $j = 1, \dots, k$ let $f^j : F \to \mathbf{L}(V^{\otimes j}, W)$ be a function which takes its values in the space of j-linear mappings from V to W. The collection ($f = f^0, f^1, \dots, f^k$)is an element of $Lip(\gamma, F)$ if the following condition holds.*

There exists a constant M such that, for each $j = 0, \dots, k$,

$$\sup_{x \in F} |f^j(x)| \leqslant M$$

and there exists a function $R_j : V \times V \to \mathbf{L}(V^{\otimes j}, W)$ such that, for each $x, y \in F$ and eachv $\in V^{\otimes j}$, we have

$$f^j(y)(v) = \sum_{l=0}^{k-j} \frac{1}{l!} f^{j+l}(x)(v \otimes (y-x)^{\otimes l}) + R_j(x, y)(v) \, ,$$

and

$$|R_j(x,y)| \leqslant M|x-y|^{\gamma-j}\,.$$

The smallest M for which the inequalities hold for all j is called the $Lip(\gamma, F)$-norm of f.

With this definition at hand, we can state the version of Picard Fix Point Theorem for differential equations driven by signals with finite p-variation for some $p < 2$.

Theorem 0.3.2. *Let p and γ be such that $1 \leqslant p < 2$ and $p \leqslant \gamma$. Assume that X has finite p-variation and that f is $Lip(\gamma)$. Then for every $\zeta \in W$, the differential equation $dY_t = f(Y_t)dX_t$, admits a unique solution, i.e. for every $\zeta \in W$ there exist a unique path $Y : [0,T] \to W$ of finite p-variation which satisfies $Y_0 = \zeta$ and*

$$Y_t = \int_0^t f(Y_s)dX_s\,.$$

In order to prove this Theorem , the technique is similar with the classical proof of the Picard Fix Point Theorem, that is we consider the map F that sends a path $Y : [0,T] \to W$ to a new path defined via $F(Y_t) = \zeta + \int_0^t f(Y_s)dX_s$ which we prove that is a contraction under suitable conditions.

0.3.1. The signature of a path. In this section, we introduce one of the fundamental ingredients of the Theory of Rough Paths, the *signature of the path*, i.e. the collection of its iterated integrals. This object appears naturally when one uses iteration in order to solve linear differential equations. Moreover, the signature of a path carries very interesting algebraic properties.

In order to simplify the computations and to introduce the signature of a path in a clear manner, we consider throughout the section the driver X to be of bounded variation. For this specific class of drivers, we consider the linear differential equation $dY_t = \bar{f}(Y_t)dX_t$, where $\bar{f} : W \to \bar{L}(V,W)$ is a linear map.

The collection of iterated integrals appears naturally when one tries to apply an iterative procedure in order to find a solution to the ordinary differential equation via fix point arguments. To make this precise, we remark that $\bar{f}(Y_t)dX_t$ can be understood in two ways. Every linear map $\bar{f} \in \mathbf{L}(W, \mathbf{L}(V,W))$ in a natural way also induces a linear map $\bar{f} : V \to \mathbf{L}(W)$, (we call it the same using a slight abuse of notation). Another abuse of notation that we use is that we have elements of $\mathbf{L}(W)$ act on W on the right instead of the left as this will simplify notation in the following.
We start with the constant path $Y_t^0 = F(Y^0)_t = Y_0(I + \bar{f}(\int_0^t dX_s))$. By re-iterating the procedure, we obtain that

$$Y_t^2 = F(Y^1)_t = Y_0 + \int_0^t \bar{f}(Y_s^1)dX_s = Y_0\left(I + \bar{f}\int_0^t dX_s + \bar{f}^{\otimes 2}\int_0^t (\int_0^s dX_u)\otimes dX_s\right)$$

After k steps, we obtain that

$$Y_t^n = Y_0\left(\sum_{k=0}^{n} \bar{f}^{\otimes k}\int_{0<u_1<\ldots<u_k<t} dX_{u_1}\otimes dX_{u_2}\ldots\otimes \dot{X}_{u_k}\right),$$

where we define $\bar{f}^{\otimes k}(x_1 \otimes \ldots \otimes x_k) = \bar{f}(x_1)\circ\ldots\circ\bar{f}(x_k)$ and then we extend linearly. In the next part of the section, we study the tensor vector spaces where this iterated integrals live and we state a result that ensures that the iterated integrals do converge.

Let $e_1, e_2, \ldots, e_d$ be a basis for V. The space $V^{\otimes k}$ is a d^k dimensional vector space with basis elements of the form $(e_{i_1} \otimes e_{i_2} \ldots \otimes e_{i_k})_{(i_1,\ldots,i_k)\in\{1,\ldots,d\}^k}$. We store the indices $(i_1, \ldots, i_k) \in \{1, 2, \ldots d\}^k$ in a multi-index I and let $e_I = e_{i_1} \otimes e_{i_2} \otimes \ldots e_{i_k}$. From the several possible norms on the tensor space $V^{\otimes k}$ (see Definition 1.25 of (insert citation) for the so-called *admissible norms*, due to its properties, we chose the *projective* or the l_1 norm defined for any element $x = \sum_{|I|=k} \lambda_I e_I \in V^{\otimes k}$ via $||x|| = \sum_{|I|=k} |\lambda_I|$. The two main properties of this norm are that for any $x \in V^{\otimes k}$ and $y \in V^{\otimes m}$ we have that $||x \otimes y|| = ||x|| ||y||$ and that for $x = \sum_{|I|=k} \lambda_I e_I$ we have $||M^{\otimes k} x|| = || \sum_{|I|=k} \lambda_I M^{\otimes k} e_I|| \leqslant \sum\limits_{|I|=k} |\lambda_I| ||M||^k = ||M||^k ||x||$.

By considering the basis of $V^{\otimes k}$ given by elements of the form e_I , we can take the dual basis in $(V^{\otimes k})^*$ as e_I^* that we can further associate the element $e_{i_1}^* \otimes e_{i_2}^* \otimes \ldots \otimes e_{i_k}^* \in (V^*)^{\otimes k}$. Furthermore, for any I, e_I^* is an element of $T(V*)$, i.e. the sub-algebra of $T((V^*))$ consisting of elements with finitely many non-zero coordinates- and running through all possible mutli indices with $|I| > 0$, we find a basis for $T(V^*)$ consisting of $e_I's$. We are now ready to introduce the following result that guarantees the convergence of the Picard iterates.

Proposition 0.3.3. *Let $X : [0, T] \to V$ be a path of bounded variation. Then it follows that*

$$|| \int_{0<u_1<\ldots u_k<t} dX_{u_1} \otimes dX_{u_2} \ldots dX_{u_k} || \leqslant \frac{||X_{[0,t]}||_{1-var}^k}{k!} .$$

0.3.2. The tensor algebra. Given that the collection of iterated integrals is such an important object in the study of the linear differential equations it is natural to consider it as a separate object that we call the signature of a path. This object takes value in the tensor algebra $T(V)$ which is described in this subsection.

The tensor algebra $T((V)) = \bigoplus_{k\geqslant 0} V^{\otimes k}$ is the infinite sum of all tensor products of V. For two elements $a = (a^0, a^1 \ldots)$ and $b = (b^0, b^1, \ldots)$ we define

$$a + b = (a^0 + b^0, a^1 + b^1, \ldots)$$

and

$$a \otimes b = (c^0, c^1, \ldots) ,$$

where $c^k = \sum\limits_{i=0}^{k} a^i \otimes b^{k-i}$. These operations together with the typical component by component multiplication by scalars $\lambda a = (\lambda a^0, \lambda a^1 \ldots)$, turn $T((V))$ into a real non-commutative algebra with unit $\mathbf{1} = (1, 0, \ldots)$. An important aspect of this Tensor Algebra is that a generic element $d \in T((V))$ is invertible if and only if $d^0 \neq 0$, and moreover there is an explicit way to compute the inverse

$$d^{-1} = \frac{1}{d_0} \sum_{k\geqslant 0} \left(\mathbf{1} - \frac{1}{d_0} d \right)^{\otimes k} . \tag{0.3.1}$$

For all $d \in T((V))$, we define the exponential and the logarithm maps via

$$\exp d = \sum_{k\geqslant 0} \frac{1}{k!} a^{\otimes k} . \tag{0.3.2}$$

$$\log d = \log d^0 + \sum_{k \geqslant 1} \frac{(-1)^k}{k} \left(\mathbf{1} - \frac{1}{d_0} d \right) , \tag{0.3.3}$$

whenever $a^0 > 0$. Like the inverse mapping, the logarithm is well defined since finitely many terms of d contribute ti each term of $\log(d)$. For the exponential map in the finite dimensional case it suffices to ensure the convergence of terms of each degree according to Lemma 2.19 of (insert citation). An important subspace of $T((V))$ that is denoted by B_0 is the is the subspace for which the term of degree zero is 0. An other important object is the multiplicative subgroup of $T((V))$ which is denoted by $T(\tilde{(V)})$, for which the term of degree zero is 1. Then the maps $\exp : B_0 \to T(\tilde{(V)})$ and $\log : T(\tilde{(V)}) \to B_0$ are bijections and inverses of each other.

0.3.3. The signature of a path, shuffle product and group-like elements. In this subsection, we introduce the signature of a path with p-variation for some $p < 2$ and describe some computation tools that are extremely useful when computing terms in the signature along with some fundamental Theorems.

Definition 0.3.4. *Let $X : [0,T] \to V$ be a path of finite p-variation for some $p < 2$. For $k \geqslant 0$, let $\int_{s<u_1<\ldots<u_k<t} dX_{u_1} \otimes \ldots \otimes dX_{u_k}$,. Then the signature of the path $X_{[s,t]}$ is given by $S(X_{[s,t]}) = (1, S(X_{[s,t]}), S(X_{[s,t]})^2, \ldots)$.*

Note that the signature of a path X lies in the subgroup $\tilde{T}((V))$ of $T((V))$. Note that the integral in the definition is a Young integral since $p < 2$. One of the properties of the signature are that is not sensitive to the starting point or to the speed that a path is traversed. That is, $S(X_{[s,t]}) = S(Y_{[u,v]})$ for $Y_w = X_{\tau w} + x$, where $\tau : [s,t] \to [u,v]$ is an increasing bijection and $x \in V$. For two paths $X_{[s,t]}$ and $Y_{[u,v]}$, we define the concatenated path $(X*Y) : [s, t+v-u] \to V$ via

$$(X * Y)_w = \begin{cases} X_w, & \text{if w} \in [s,t] \\ X_t + Y_w - Y_t, & \text{if w} \in [t, t+v-u] . \end{cases}$$

In this setting, we have one result about how does the signature map behave for the concatenated path in terms of the paths, that is attributed to Chen.

Theorem 0.3.5. *For two paths $X_{[s,t]}$ and $Y_{[u,v]}$ of finite p-variation for some $p < 2$, we have that*

$$S(X * Y) = S(X) \otimes S(Y) .$$

Chen's Theorem also gives that the range of the signature is closed in $T((V))$. Moreover, if we consider the reversal of the path $X : [s,t] \to E$ given by $X^r : [s,t] \to E$ given by $X^r_u = X_{s+t-u}$, then we have

Proposition 0.3.6. *Let $X : [s,t] \to V$ be a path of finite p-variation for some $p < 2$, and $X^r_{[s,t]}$ be its reversal. Then it follows that $S(X^r) = S(X)^{-1}$.*

Using the previous Proposition and the fact that $S(x_{[s,t]}) = \mathbf{1}$ with $x \in V$ ($x_{[s,t]}$ is understood as the constant path at x,) we conclude that the range of the signature map forms a multiplicative subgroup of $\tilde{T}((V))$.

We now define another useful property when it comes to dealing with signatures of paths. For $i, j \in \mathbb{Z}$, we define the *shuffle product* $Sh(i,j) \subset S_{i+j}$ as the permutations σ on the set $\{1, 2, \ldots, i+j\}$ which preserve the order of the sets $\{1, 2, \ldots, i\}$ and $\{i+1, \ldots, i+j\}$, that is

$\sigma(1) < \ldots < \sigma(i)$ and $\sigma(i+1) < \ldots < \sigma(i+j)$. For two multi-indexes $U = (u_1, \ldots, u_i)$,and $V = (v_1, \ldots, v_j)$, we define the joint multi-index $H = (h_1, \ldots, h_{i+j}) = (u_1, \ldots, u_i, v_1, \ldots, v_j)$, and for $\sigma \in S_{i+j}$ we define $\sigma(H) = (h_{\sigma_1}, \ldots, h_{\sigma(i+j)}$. Then, we define for e_U^* and $e_V^* \in T(V*)$ the shuffle product by

$$e_U^* \sqcup e_V^* = \sum_{\sigma \in Sh(i+j)} e^*_{\sigma^{-1}(H)} \in T(V^*)\,.$$

We further define the *group-like elements* of $\tilde{T}((V))$, denoted by $G^{(*)}$, consisting of all elements $a \in \tilde{T}((V))$ for which $f(a)g(a) = (f \sqcup g)(a) \quad \forall f, g \in T(V^*)$.

The verification that $G^{(*)}$ forms indeed a subgroup of $\tilde{T}((V))$ is proved in Lemma 2.17 in [5]. One of the most important results about this group is that is the subspace where the signature actually lives.

Theorem 0.3.7. *For a path $X_{[s,t]}$ of finite p-variation with $p < 2$, it holds that $S(X) \in G^{(*)}$.*

We put on $T((V))$ the product topology (i.e. the topology for which a sequence of elements $(a_n)_{n \geqslant 0} \in T((V))$ converges to $a \in T((V))$ in the product topology if and only if it converges coordinate-wise, i.e. for every $k \geqslant 0$, the limit $\lim_{n\to\infty} a_n^k = a^k$. Also, all the maps and operations discussed so far, that is $+, \otimes : T((V)) \times T((V)) \to T((V))$, inversion, exp and log are continuous with respect to the product topology on $T((V)) \times T((V)$.

The (associative) tensor algebra $T((V))$ can be viewed as a Lie algebra with the Lie bracket given by $[a, b] = a \otimes b - b \otimes a$. When studying the Lie algebra $T((V))$, two Lie subalgebras are of interest. First the Lie sub algebra generated by the elements $V \subset T((V))$, which we denote by $\mathcal{L}(V)$, and $\mathcal{L}((V))$ that is the completion of $\mathcal{L}(V)$ in $T((V))$ with respect to the product topology. In order to describe $\mathcal{L}(V)$ and $\mathcal{L}((V))$, we have the following procedure. For the subspace $U, W \subset T((V))$, we define $[U, W] \subset T((V))$ as the subspace spanned by all the elements of the form $[a, b]$ with $a \in U$ and $b \in W$. We define inductively for $k \in \mathbb{N}$, $L_k = [V, L_{k-1}] \subset V^{\otimes k}$, with $L_0 = 0$ and $L_1 = V$. It follows that

$$\mathcal{L}((V)) = \bigoplus_{k \geqslant 0} L_k$$

and naturally $\mathcal{L}(V)$ consists of those elements of $\mathcal{L}((V))$ which have finitely many non-zero terms. The elements of $\mathcal{L}((V))$ are typically called Lie series and those of $\mathcal{L}(V)$ Lie polynomials. As a complete classification of group-like elements, we have the following result

Theorem 0.3.8. *An element is group-like if and only if its logarithm is a Lie series. That is* $\log G^{(*)} = \mathcal{L}((V))$.

Free Nilpotent Lie Group and Rough Paths

All the spaces that we defined so far in our Analysis, have also a finite dimensional version (that we call the truncated version) which consists of terms up to a fixed degree n. In order to define them precisely, we consider

$$B_n = \{a = (a^0, a^1, \ldots) | a^0 = a^1 = \ldots = a^n = 0\}$$

that is an ideal of $T((V))$. We define the truncated tensor algebra $T^{(n)}(V) = T((V))/B_n$ with $\pi_n : T((V)) \to T^{(n)}(V)$ the natural quotient map. As an algebraic structure, $T^{(n)}(V)$ is an algebra where multiplication $\otimes$ is the same as in $T((V))$ but truncated at level n (i.e. all the elements with degree higher than n are automatically null in this space). Between the truncated spaces , we can define truncated versions of the log map or of the exponential exp map. We also can view $T^{(n)}(V) = \bigoplus_{0\leqslant k\leqslant n} V^{\otimes k}$ - as a finite sum of finite dimensional vector spaces. Thus, when V is of dimension d, $T^{(n)}(V)$ is of dimension $1+d+\ldots+d^n$. We remark also that we consider $T^{(n)}(V)$ endowed with the product topology from $T((V))$. In this setting, as before, the operations remain continuous.

We consider also the sets of truncated group-like elements and Lie polynomials of degree n, given by $G^{(n)} = \pi_n G^{(*)}$ and $\mathcal{L}^{(n)}(V) = \pi_n\mathcal{L}(V) = \pi_n\mathcal{L}((V))$. Also, remark that $\tilde{T}^{(n)}(V) = \pi_n\tilde{T}((V))$ is a simply connected n-step nilpotent Lie group with Lie algebra $\pi_n B_0$ for which $\exp^{(n)} : \pi_n B_0 \to \tilde{T}^{(n)}(V)$ is a diffeomorphism. We have also that

Proposition 0.4.1. *An element $a \in T((V))$ is group-like if and only if $\pi_n a \in G^{(n)}$ for all $n \geqslant 0$ if and only if $\log^{(n)} \pi_n a \in \mathcal{L}^{(n)}(V)$ for all $n \geqslant 0$.Furthermore, $G^{(n)}$ is a closed , simply connected n-step nilpotent Lie subgroup of $\tilde{T}^{(n)}(V)$, for which $\mathcal{L}^{(n)}(V)$ is the Lie algebra and $\exp^{(n)}$: $\mathcal{L}^{(n)} \to G^{(n)}$ is the diffeomorphism.*

We call $G^{(n)}$and $\mathcal{L}((V))$ the free-step nilpotent Lie group and algebra over V. Also, we can consider naturally the notion of truncated signature $S^{(n)}(X_{[s,t]}) = \pi_n S^{(n)}(X_{[s,t]})$ as an element of $G^{(n)}$. We also have that for two paths X and Y, $S^{(n)}(X*Y) = S^{(n)}(X)\otimes S^{(n)}(Y)$ and $S^{(n)}(X)^{-1} = S^{(n)}(X^r)$, with X^r the reversal of X. An important result that gives a description of elements in $G^{(n)}$ is the Chow-Rashevskii Theorem.

Theorem 0.4.2. *For every $a \in G^{(n)}$ there exists a piecewise linear path $X : [0,T] \to V$ such that $a = S^{(n)}(X_{[0,T]})$.*

Remarkably, each element of $G^{(n)}$ is in fact the signature of some piecewise linear path. With this result at hand, we can define the Carnot-Caratheodory (CC) norm on the group $G^{(n)}$ by $||a||_{CC} = \inf_{S^{(}n)(X_{[0,T]})|} ||X||_{1-var}$ which is guaranteed to be finite. Since $G^{(n)}$ is not a vector space, the CC norm is not a norm in the real sens but gives rise to a metric on $G^{(n)}$ by $d_{CC}(a,b) = ||a^{-1}b||_{CC}$. We are now ready to define the notion of *p-rough path*. This notion builds naturally on previous results. These paths are taking values in the Lie group $T^{(n)}(V)$ and satisfy an algebraic condition- Chen's identity- and an analytic bound that is very similar with the factorial decay of the norms of the levels of signature presented in Proposition (insert citation). We now define the notion of *multiplicative functional*

Definition 0.4.3. *Let $n \geqslant 1$ be an integer and let $X\Delta_T \to T^{(n)}(V)$ be a continuous map. Denote with $X_{s,t}$ the image of the interval (s,t) by X, and write*

$$X_{s,t} = (X^0_{s,t}, \ldots X^n_{s,t}) \in \mathbb{R} \oplus V \oplus V^{\otimes 2} \ldots \oplus V^{\otimes n}.$$

The function X is called multiplicative functional of degree n in V if $X^0_{s,t} = 1$ and for all $(s,t) \in \Delta_t$ we have

$$X_{s,u} \otimes X_{u,t} = X_{s,t} \;\; \forall s,u,t \in [0,T].$$

The multiplicative property is called $Chen'sidentity$ (see connection with Theorem-Chen .) To get familiar with the object, let us take the following example

Example 0.4.4. Let X be a multiplicative functional of degree 1 that is for all $(s,t) \in \Delta_T, X_{s,t} \in T^{(1)}(V)\mathbb{R} \oplus V$. That is $X_{s,t} = (1, X^1_{s,t})$, with $X^1_{s,t} \in V$. Then Chen's identity reads

$$(1, X^1_{s,t}) = (1, X^1_{s,u}) \otimes (1, X^1_{u,t}) = (1, X^1_{s,u} + X^1_{u,t}).$$

So, in $T^{(1)}(V)$ (at 'first level') the multiplication condition in $T^{(1)}(V)$ reduces to the additivity in V of the mapping $(s,t) \to X^1_{s,t}$. This fact is equivalent with the existence of a path $X^1_{s,t} = x_t - x_s$, for all $(s,t) \in \Delta_T$. This path is unique up to an addition of a constant element in V. Note that if X is a multiplicative functional of degree $n \geqslant 2$ in V then we have that $\pi_1 X : \Delta_T \to T^{(1)}(V)$ is a multiplicative functional of degree 1.This implies that there exists a classical path $x : [0,T] \to V$, which underlies X (i.e. $X^1_{s,t} = x_t - x_s$,) but X is not the signature of X in general. First of all X may have p-variation for some $p \geqslant 2$ in which case the signature does not exist. Even if the signature does indeed exist, consider for example the path $X_{s,t} = (1, 0, (t-s)w)$ that is a multiplicative functional of degree 2. If we look only at the first level than the underlying path x is a constant path whose signature is simply $(1,0,0)$ which is different from X. We are now ready to define the central object of Rough Paths Theory.

Definition 0.4.5. *A $p-$rough path of degree n is a map $X : \Delta_T \to \tilde{T}^{(n)}(V)$ which satisfies Chen's identity $X_{s,t} \otimes X_{t,u} = X_{s,u}$ and the following 'level dependent' analytic bound*

$$||X^i_{s,t}|| \leqslant \frac{w(s,t)^{\frac{i}{p}}}{\beta_p (\frac{i}{p})!},$$

where $y! = \Gamma(y+1)$ whenever y is a positive real number and β_p, is a positive constant.

We call w a $p-$ variation control of X. The factors $(\frac{i}{p})!$ and β_p in the definition of $p-$ rough path are not important due to the possibility of recalling; however, they become important in the following Extension Theorem that stat that there is only one way to extend a $p-$ rough path to the entire group $\tilde{T}((V))$.

Theorem 0.4.6. *Let $p \geqslant 1$ be a real number and $n \geqslant 1$ be an integer. Let $X : \Delta_T \to T^{(n)}(V)$ be a multiplicative functional with finite $p-$variation controlled by w. Assume that $n \geqslant [p]$. Then there exists a unique extension of X to a multiplicative functional $\Delta_T \to T((V))$ which has finite $p-$ variation, i.e. for every $m \geqslant [p]+1$, there exists a unique continuous function $X^m : \Delta_T \to V^{\otimes m}$ such that $(s,t) \to X_{s,t} = (1, X^1_{s,t}, \ldots, X^{[p]}_{s,t}, \ldots, X^m_{s,t}, \ldots) \in T((V))$, is a multiplicative functional with finite $p-$variation controlled by w, in the sense of definition 0.4.5, with $\beta_p = p^2 \left(1 + \sum_{r=3}^{\infty} \left(\frac{2}{r-2}\right)^{\frac{[p]+1}{p}}\right)$.*

We denote the space of all $p-$rough paths of degree $[p]$ by $\Lambda_p(V)$. A first remark is that for every continuous map $X : \Delta_T \to \tilde{T}((V))$, we associate the path $x_t = X_{0,t}$, and conversely for every $x \in C([0,T], \tilde{T}^{(n)}(V))$ we can define the mapping $X : \Delta_T \to \tilde{T}((V))$ by $X_{s,t} = x_s^{-1} x_t$, which indeed satisfies Chen's identity. Thus, every map $X : \Delta_T \to \tilde{T}((V))$ that satisfies Chen's identity is completely characterized by its associated path $x : t \to X_{0,t}$. For a path with $p-$ variation smaller than 2, that is canonically a $p-$rough path of degree $[p] = 1$, we have the signature $S(X) : (s,t) \to S(X_{[s,t]})$, that satisfies Chen's identity also. Furthermore, a modification of the factorial decay of the signature Proposition gives that the signature satisfies the required analytic bound also. To sum up, we are in the position to apply Theorem 0.4.6 to obtain that the signature is the unique extension of a $p-$ rough path for $p < 2$. Also, the Extension map is a continuous mapping as stated in the following important result.

Theorem 0.4.7. *Let $X, Y \in \Lambda_p(V)$ with $p-$variation control w and let $\varepsilon \in (0,1)$. Then the bound*

$$||X^i_{s,t} - Y^i_{s,t}|| \leqslant \varepsilon \frac{w(s,t)^{\frac{i}{p}}}{\beta_p(\frac{i}{p})!}\,,$$

holds for all $(s,t) \in \Delta_t$ and $i\ geq 1$, provided that it holds for all $1 \leqslant i \leqslant [p]$.

The spaces of p-Rough Paths and Geometric Rough Paths

We introduce a metric on $\Lambda_p(V)$ which transform the space $\Lambda_p(V)$ in a complete metric space. For $X, Y \in \Lambda_p(V)$ we define

$$d_p(X,Y) = \max_{1\leqslant i\leqslant [p]} \sup_{\mathcal{D}\subset[0,T]} \left(\sum_{\mathcal{D}} ||X^i_{t_i,t_{i+1}} - Y^i_{t_i,t_{i+1}}||^{\frac{p}{i}} \right)^{\frac{i}{p}}\,.$$

In general, this quantity might be infinite for general mappings from $\Delta_t \to T((V))$, but in the context of $p-$rough paths this is finite due to Theorem 0.4.7. Related with this notion is a notion of convergence (stronger notion) that is the *convergence in the p-variation topology*. Formally, this is defined in terms of converging sequences. A sequence $(X(n))_{n\geqslant 1} \in \Lambda_p(V)$ is said to converge to $X \in \Lambda_p(V)$ in $p-$ variation topology if there exists a $p-$ control w of X and $X(n)$ for all $n \geqslant 1$, and a sequence $(a(n))_{ngeq1}$ of positive reals such that $\lim_{n\to\infty} a(n) = 0$ and

$$||X(n)^i_{s,t} - X^i_{s,t}|| \leqslant a(n) w(s,t)^{\frac{i}{p}}\,,$$

for all $(s,t) \in \Delta_T$ and $1 \leqslant i \leqslant [p]$.

A path in V with finite $q-$ variation for some $q < 2$ defines q-rough path. For $p \geqslant q$ this $q-$rough path can be extended by the Extension Theorem to a multiplicative functional of degree $[p]$ with finite $q-$ variation, hence finite $p-$variation. This means that in particular, a path with bounded variation can be considered canonically as a $p-$rough path for every $p \geqslant 1$, where the extension is given by the signature as discussed before. We are now ready to define the notion of a geometric rough paths.

Definition 0.5.1. *A geometric $p-$ rough path is a $p-$ rough path that can be expressed as a limit of $1-$rough paths in the $p-$ variation metric.*

The space of geometric rough paths in V is denoted by $G\Omega_p(V)$.

Example 0.5.2. Let us take $V = \mathbb{R}^d$, $(d > 2)$. Let $(x^i)_{i=1,\dots,d}$ be an element of V. The $2-$tensors are pictured as arrays $(x^{ij})_{i,j\in 1,\dots,d}$. Let us take x_u to be a path of finite $1-$ variation in $\mathbb{R}^d$. We define $X^i_{s,t} = x^i_t - x^i_s$, and $X^{i,j}_{s,t} = \int\int_{s<u_1<u_2<t} dx^i_{u_1} dx^j_{u_2}$. Then $X_{s,t} = (1, (X^i_{s,t})^d_{i=1}, (X^{i,j}_{s,t})^d_{i,j=1})$, is a multiplicative functional in $T^{(2)}(\mathbb{R}^d)$, with finite $p-$ variation for some $p < 3$. This functional belongs to $G\Omega_p(\mathbb{R}^d)$ and is called the canonical extension of the path x_u. Writing $(X^{i,j}_{s,t})^d_{i,j=1}$ into its symmetric and anti-symmetric parts, we obtain

$$X^{i,j}_{s,t} = \frac{1}{2}(x^i_t - x^i_s)(x^j_t - x^j_s) + A^{i,j}_{s,t}\,,$$

where $A^{i,j}_{s,t} = \frac{1}{2}\int\int_{s<u_1<u_2<t} dx^i_{u_1} dx^j_{u_2} - dx^j_{u_1} dx^i_{u_2}$. The $A^{i,j}_{s,t}$ has the geometric interpretation of the area enclosed by the path concatenated with the chord between the points given by the indices i and j. Thus by integrating the winding number over the plane gives $A^{i,j}_{s,t}$.

0.5.1. Rough Differential Equations (RDE) and Rough Differential equations with drift. Throughout this section we work with a suitable space and a suitable notion of distance needed in order to set up exactly the definitions for Rough Differential Equations (RDE's).

Definition 0.5.3. *A weakly geometric $p-$rough path is $p-$ rough path which takes its values in $G^{[p]}$, the free nilpotent group of step $[p]$. The space of weakly geometric rough paths is denoted by $WG\Omega_p(V)$.*

This space is indeed related with the space of geometric rough path via the inclusion $G\Omega_p(V) \subset WG\Omega_p(V)$, and this inclusion is strict. For futher details, see [5], 3.22. Also, we work with the notion of distance given by $d_{0;[0,1]}(\mathbf{x}, \mathbf{y}) := \sup_{0 \leqslant s<t \leqslant 1} d(x_{s,t}, y_{s,t})$. Rough Differential Equations introduction with all the detailes needed can be found in Chapter 10 in [2]. For now let $x \in C^{1-var}([0,1], \mathbb{R}^d)$. For the solution y to the following controlled differential equations

$$dy = V(y)dx := \sum_{i=1}^{d} V_i(y)dx^i, \quad y_0 \in \mathbb{R}^q,$$

we use the notation $\pi_(V)(0; y_0, x)$. In fact, the notation $\pi_{(V)}(s, y_s; x)$ stands for solutions of controlled ODE with vector fields $V = (V_i)_{i=1,\dots,d}$ started at time s from a point $y_S \in \mathbb{R}^q$.

Definition 0.5.4. *Let $\mathbf{x} \in WG\Omega_p(V)$ for some $p \geqslant 1$. We say that $y \in C([0,1], W)$ is a solution to the rough differential equation driven by the signal $\mathbf{x}$ along the collection of vector fields $V = (V_i)_{i=1,\dots,d}$ and started at y_0 if there exists a sequence $(x_n)_n$ in $\mathbb{C}^{1-var}[0,1], W)$ such that*

- $\lim_{n \to +\infty} d_{0;[0,1]}(S_{[p]}(x_n), \mathbf{x}) = 0$, *where $S_{[p]}(x_n)$ stands for the truncated signature of x_n at level $[p]$.*
- $\sup_n ||S_{[p]}(x_n)||_{p-var;[0,1]} \leqslant \infty$;

and ODE solutions $y_n = \pi_{(V)}(0, y_0; x_n)$ such that

$$y_n \to y \text{ uniformly on [0,1] as} n \to \infty$$

This situation is denoted with the formal equation:

$$dY = V(y)d\mathbf{x}, y_0 \in \mathbb{R}^q,$$

which is called rough differential equation.

The solution map of a *RDE*

$$\mathbf{x} \in C^{p-var}([0,1], G^{[p]}(V)) \to \pi_{(V)}(0, y_0; \mathbf{x}) \in C([0,1], G^{(p)}(\mathbb{R}^q).$$

is known in the Rough Paths Literature as the *Itô* map.

We extend our study to the Rough Differential equations with drift term as this fits perfectly with the study of the Loewner differential equation as a Rough Differential Equation. We start with the corresponding definition of RDE's with drift.

Definition 0.5.5. *Let $p, q \geqslant 1$ and such that $1/p + 1/q > 1$. Let $\mathbf{x} \in C^{p-var}([0,T], G^{[p]}(\mathbb{R}^d))$ be a weak geometric $p-$ rough path and $\mathbf{h} \in C^{q-var}([0,T], G^{[p]}(\mathbb{R}^{d'}))$ be a weakly geometric $q-$ rough path. We say that $y \in C([0,T], \mathbb{R}^e)$ is a a solution to the rough differential equation with drift driven by (x,h) along the collection of $\mathbb{R}^e$ vector fields $((V_i)_{1 \leqslant i \leqslant d}, (W_j)_{1 \leqslant j \leqslant d'})$ and started at y_0 if there exists a sequence $(x^n, h^n) \subset C^{1-var}([0,T], \mathbb{R}^d) \times C^{1-var}([0,T], \mathbb{R}^{d'})$, such that*

- $\sup_n ||S_{[p]}(x_n)||_{p-var;[0,T]} + ||S_{[p]}(h_n)||_{q-var;[0,T]} \leqslant \infty$.
- $\lim_{n\to+\infty} d_{0;[0,T]}(S_{[p]}(x_n), \mathbf{x}) = 0$, *and* $\lim_{n\to+\infty} d_{0;[0,T]}(S_{[p]}(h_n), \mathbf{h}) = 0$,

and ODE solutions $y_n = \pi_{(V)}(0, y_0; (x_n, h_n))$ *such that*

$$y_n \to y \text{ uniformly on [0,1] as} n \to \infty$$

This situation is denoted with the formal equation:

$$dY = V(y)d\mathbf{x} + W(y)d\mathbf{h}\,, y_0 \in \mathbb{R}^q\,,$$

which is called rough differential equation with drift .

A very imporatant set of Theorems regarding RDE's with drift are the ones that give existence , uniquness of solutions and continuity estimates in terms of the regularity of the vector fields involved in the RDE with drift. The Theorems that follow are in the spirit of the *Universal Limit Theorem* from [5] .

The first theorem, is giving existence of solution along with some continuity estimates. However this result does not guarantee uniqueness of solution.

Theorem 0.5.6. *Let* $V = (V_i)_{1\leqslant i\leqslant d}$ *a collection of vector fields in* $Lip^{\gamma-1}(\mathbb{R}^d)$ *with* $\gamma \geqslant 1$, *and* $W = (W_j)_{1\leqslant i\leqslant d'}$ *a collection of vector fields in* $Lip^{\beta-1}(\mathbb{R}^d)$ *with* $\beta > 1$. *Let* $x, \tilde{x}$ *be two paths in* $C^{1-var}([s,u], \mathbb{R}^d)$ *such that* $S_{[\gamma]}(x)_{s,u} = S_{[\gamma]}(\tilde{x})_{s,u}$, *and* $h, \tilde{h}$ *be two paths in* $C^{1-var}([s,u], \mathbb{R}^d)$ *such that* $S_{[\beta]}(h)_{s,u} = S_{[\beta]}(\tilde{h})_{s,u}$. *We then have that the two corresponding RDE's with drift have solutions and moreover we have the following estimate for some* $C = C(\gamma, \beta)$,

$$|\pi_{V,W}(s, y_s; (x,h))_{s,u} - \pi_{V,W}(s, y_s; (\tilde{x}, \tilde{h}))_{s,u}| \leqslant C(l_h^\beta + l_x l_h^{\gamma-1} + l_x l_h + l_x^{\beta-1} l_h + l_x^\gamma) exp(C(l_x + l_h))\,.$$

where l_x *and* l_h *are bounds for* $|V|_{Lip^{\gamma-1}} \max\{\int_s^u |dx|, \int_s^u |d\tilde{x}|\}$ *respectively* $|W|_{Lip^{\beta-1}} \max\{\int_s^u |dh|, \int_s^u |d\tilde{h}|\}$.

In order to provide existence and uniqness of solution, the collection of vector fields defining the RDE with drift should be in Lip_γ respectively Lip_β. See Theorem 12.10 and Theorem 12.11 in [2] . So, the fundamental difference between the two results is in the regularity of the vector fields that give the transition between existence and uniqueness and guaranteed existence only. In the classical theory of ordinary differential equations this results are in the spirit of $Cauchy - Peano$ Theorem and Picard-Lindelhöf Theorem , i,e, in the transition between just continuous vector field versus Lipschitz vector field in the space variable.We would like to make this transition of phase uniqueness/non-uniqueness of solution to an RDE apparent also in the Stochastic version of Loewner differential equation.

Results about SLE using Rough Paths Theory

0.6.1. Expected signature for chordal SLE_κ for $\kappa \in [0,4]$. In this section we present the result of Prof. Brent Werness that uses the left-passage probability of the SLE trace and a version of Green's Theorem to compute several terms in the expected signature of the SLE_κ , for $\kappa \in [0,4]$. The out-line of the paper is as follows In order to prove that for $\kappa \leqslant 4$, SLE has finite p-variation for some $p < 2$ (p-variation is related with Holder regularity as mentioned in the Introduction to Rough Paths Chapter), they used a Theorem of Aizenman and Burchard that can be found in (insert citation) stating that : If you can bound the probability that your random curve crosses annulus at least k times, then you have the equality between upper box dimension and the inverse of the supremum coefficient α for which you can reparametrize your curve to be β Hölder, for $\beta < \alpha$.

The upper box dimension for SLE can be computed and it is proved in the paper that this has the value $1 + \kappa/8$ with probability 1. The main insight provided by this paper is that SLE respects this annular crossing property and thus the Theorem of Ainzemnann and Burchard applies for this random curves. Using this Theorem of they concluded from the equality between upper box dimension and the inverse of supremum Hölder coefficient for SLE that SLE has indeed finite p-variation for some $p < 2$.

In order to show that the result of Ainzemann and Burchard about random curves is applicable for SLE we need to show that the probability that SLE crosses annulus at least k times is bounded in a certain way, i.e. an uniform estimate on the probability that the SLE_κ crosses an annulus at least k times. In order to bound this probability, throughout the paper is develop an abstract framework to discuss this situation. To relate these abstract results that will not constitute the object of our exposure, with probabilities involving $SLE's$ they used Lemma 5.3 page 19 -result using excursion measure to bound the probability that the SLE path intersects a curve-. Finally, in the paper is provided a bound for the excursion measure in annular domain using Beurling estimate (very briefly explained on page 19 after the Lemma also).
The output of this is that the chordal SLE_κ in the unit disk for $\kappa \leqslant 4$ can be reparametrized to be Hölder continuous of any order up to $\frac{1}{1+\kappa/8}$. The Corollary of this result is that the the notion of Young integration for the SLE_κ is well-defined with probability one. Thus, there exists a pathwise notion of integration for the SLE_κ in the regime $\kappa \in [0,4]$. A consequence of this Corollary is that the notion of signature is also well defined for SLE_κ for $\kappa \in [0,4]$.

The rest of the section is devoted for a detailed explanation of the paper focusing on the SLE estimates and the new ideas, rather than on the abstract machinery presented in the second part of the paper in order to create the framework to discuss the annular crossings for random curves.

In order to define SLE , we paramatrize the curve by the half-plane capacity (i.e. the capacity of the curve -that is the coefficient of the $\frac{1}{z}$ in the expansion at ∞ of the corresponding conformal map- increases linearly and deterministic in time and because of this we choose it to be $2t$,). In this parametrization is proved that several planar statistical physics models such as Ising interfaces, Loop erased random walks etc. converge to versions of SLE_κ with corresponding parameters ($\kappa = 3$, for Ising interfaces and $\kappa = 2$ for loop erased random walk) . However, by doing so, we loose the information about the regularity of these parametrizations. The main motivation for the paper of Prof. Brent Werness is what are the best regularity properties that SLE_κ curves can have under *arbitrary* reparametrization?

For example, the regularity of SLE_κ under capacity parametrization is well understood, in the sense that Viklund and Lawler proved in (insert citation) that the optimal Hölder exponent is

$$\alpha_0 = \min\{\frac{1}{2}, 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8+\kappa}}\}\,.$$

The discrepancy comes from the fact that Beffara proved in (include citation) that the almost sure Hausdorff dimension of a chordal SLE_κ, is $1 + \frac{\kappa}{8}$, and a $d-$dimensional curve γ can not be reparametrized to be Hölder continuous of any order greater than $1/d$. However, the capacity parametrization does not give too much insight about the possibility of reparametrizing the SLE curve of any of the remaining orders up to $\frac{1}{d}$. Prof. Werness answers this question and the answer is summarized in the following Theorem

Theorem 0.6.1. *Fix $0 \leqslant \kappa \leqslant 4$ and let $\gamma : [0, \infty] \to \mathbb{D}$, be a chordal SLE_κ from -1 to 1 in $\mathbb{D}$ and $d = 1 + \frac{\kappa}{8}$ be its almost sure Hausdorff dimension. Then, a.s. we have that*

- *for any $\alpha < 1/d$, γ can be reparametrized as a curve $\tilde{\gamma} : [0, 1] \to D$ which is Hölder continuous of order α.*
- *for any $\alpha > 1/d$, γ can not be reparametrized as a curve $\tilde{\gamma} : [0, 1] \to D$ which is Hölder continuous of order α.*

The main result used to prove this Theorem is the result of Ainzemnann and Burchard mentioned in the overview. To obtain the uniform estimate on the probabilities that SLE crosses an annulus k times that is needed in order to apply the result of Ainzenmann and Burchard, we have that for $A_r^R(z_0)$- the annulus with inner radius r and outer radius R centered at z_0 that

Lemma 0.6.2. *Fix $\kappa \leqslant 4$, and let $\beta = 8/\kappa - 1$. For any $k \geqslant 1$, there exists c_k so that for any $z_0 \in \mathbb{D}, r < R$, we have that*

$$\mathbb{P}[\gamma\ traverses\ A_r^R(z_0)\ at\ least\ k\ different\ times] \leqslant c_k \left(\frac{r}{R}\right)^{\beta/2([k/2]-1)}\,.$$

We now introduce the objects and definitions that are introduced throughout the paper. Firstly, we are working with SLE_κ from -1 to 1 in $\mathbb{D}$. Secondly, we need to introduce various definitions that we are using throughout the proof.
Let

$$\alpha(\gamma) = \sup\{\alpha | \gamma \text{ admits an } \alpha - \text{Hölder reparametrization}\}\,.$$

Another similar notion is the *turtuosity*. Let $M(\gamma, l)$ be the minimal number of segments needed to partition the curve γ into segments no greater than l. We would like to understand the power law growth of this concept and for this we define the *turtuosity exponent*

$$\tau(\gamma) = \inf\{s > 0 | l^s M(\gamma, l) \to 0, \text{as l} \to 0\}\,.$$

These two notions are connected via the following result

Theorem 0.6.3. *For any curve $\gamma : [0, 1] \to \mathbb{R}^d$,*

$$\tau(\gamma) = \alpha(\gamma)^{-1}\,.$$

Furthermore, we consider $N(\gamma, l)$ to be the minimal number of sets of diameter at most l needed to cover γ. We define the *upper box dimension* by

$$dim_B(\gamma) = \inf\{s > 0 | l^s N(\gamma, l) \to 0, \text{as l} \to 0\}.$$

Note that the two notions can differ significantly if we consider a random curve that wiggles in small ball of radius ε. What is desired is actually a condition which detereminstically ensures that the upper box dimension and the turtuosity exponent coincide such as the optimal Hölder exponent can be controlled via the upper box dimension. The paper of Ainzenmann and Burchard provide such a property which is referred to as *tempered crossing property*. We say that a curve γ exhibits a k-fold crossing of power ε at the scale $r < 1$ if it traverses a spherical shell of the form

$$D(x; r^{1+\varepsilon}, r) := \{y \in \mathbb{R}^d | r^{1+\varepsilon} \leqslant |y - x| \leqslant r\}.$$

We say that a curve γ has the tempered crossing property if for every $0 < \varepsilon < 1$ there exists $k(\varepsilon)$ and $0 < r_0(\varepsilon) < 1$ such that on scales smaller than $r_0(\varepsilon)$ the curve has no $k(\varepsilon)$ crossings of power ε.

We use the following result from (insert citation)

Theorem 0.6.4. *Let $\gamma : [0, 1] \to \Lambda$ be a random curve contained in some compact set in Λ. If for all k there exist c_k and $\lambda(k)$ so that for all $x \in \Lambda$ and all $0 \leqslant r \leqslant R \leqslant 1$ we have*

$$\mathbb{P}[\gamma \ \textit{traverses D(x; r, R)at least k separate times}] \leqslant c_k \left(\frac{r}{R}\right)^{\lambda(k)},$$

where $\lambda(k) \to \infty$ as $k \to \infty$ then the tempered crossing probability holds almost surely and

$$dim_B(\gamma) = \tau(\gamma) = \alpha(\gamma)^{-1}.$$

To prove that SLE_κ for $\kappa \in [0, 4]$ satisfies the conditions of 0.6.4 is the content of the Section 5 of the paper and it not the object of this essay. By applying Theorem 0.6.4, we obtain that for SLE_κ with $\kappa \in [0, 4]$, we have that $dim_B(\gamma) = \alpha(\gamma)^{-1}$. For SLE the value of the box dimension is know to be $1 + \kappa/8$, with probability one. In order to prove this, we use that the result in [6] Section 8 that proves that the upper box dimension is bounded by above by $1 + \kappa/8$, along with the fact that the Hausdorff dimension (that is proven to be $1 + \kappa/8$ a.s. by Beffara) is a lower bound for the box dimension.

We would like to compute few gradings in the *expected signature* of SLE. In order to simplify the computation, we work with chordal SLE_κ defined from 0 to 1 in the disk of radius $1/2$.

Let $\gamma_1, \gamma_2 : [0, 1] \to \mathbb{R}$ denote the real and imaginary components of $\gamma : [0, 1] \to \mathbb{C}$. We define the coordinate integrals as

$$\gamma_{k_1 k_2 \ldots k_n} := \int_{0 < t_1 < t_2 < \ldots < t_n < 1} d\gamma_{k_1}(t_1) d\gamma_{k_2}(t_2) \ldots d\gamma_{k_n}(t_n).$$

It is convenient to define the multi-index $\mathbf{k} = k_1 k_2 \ldots k_n$. When computing the gradients, we use the notion of *shuffle product* introduced in the Introduction to Rough Paths section.

Let $e_{\mathbf{k}} = e_1 \otimes e_2 \ldots \otimes e_{k_n}$ be the basis element for the formal series of tensors of the standard basis $\mathbb{R}^2$ (viewed as $\mathbb{C}$). The signature of γ is defined to be

$$S(\gamma) = \sum_{\mathbf{k}} \gamma^{\mathbf{k}} e_{\mathbf{k}} \, ,$$

and by taking the expectation, we obtain that

$$\mathbb{E}[S(\gamma)] = \sum_{\mathbf{k}} \mathbb{E}[\gamma^{\mathbf{k}}] e_{\mathbf{k}} \, ,$$

The computation of expected signature uses the left-passage probability that was computed in the previous section. Due to the fact that SLE is a conformally invariant object, by mapping the upper-halfplane into the unit disk (and after that by simply shifting, to our disk), the left passage probability is related with the probability (that we will denote by $p(x, y)$) that the point $z = x + iy$ is situated above or below the chordal SLE_κ from 0 to 1 .The conformal map that maps D to $\mathbb{H}$ is $z \to \frac{iz}{1-z}$. The first three grading in the expected signature are computed in the following Theorem.

Theorem 0.6.5. *Fix $\kappa \leqslant 4$ and let*

$$\begin{aligned} A_\kappa &= \frac{1}{12} - \int_D y p(x, y) dx dy \\ &= \frac{C_k}{4} \left[\int_0^{\pi/2} \frac{sin(t) - tcos(t)}{sin^3(t)} cos^\lambda(t) dt \right] - \frac{1}{24} \, , \end{aligned}$$

where $C_k^{-1} := \int_0^\pi sin^\lambda(t) dt$ and $\lambda = \beta - 1 = 8/\kappa - 2$. Then,

$$\mathbb{E}[S(\gamma)] = 1 + e_{\mathbf{1}} + \frac{1}{2} e_{\mathbf{11}} + \frac{1}{6} e_{\mathbf{111}} + A_\kappa e_{\mathbf{122}} - 2A_\kappa e_{\mathbf{212}} + A_\kappa e_{\mathbf{221}} + \dots$$

Since we proved that SLE_κ has finite p-variation for some $p < 2$, then the signature exist and is well defined for SLE for $\kappa \in [0, 4]$. The first entry in the signature is 1 by convention. Inspecting the first level in the signature (i.e. the increment of the paths) , we obtain for SLE in the disk from 0 to 1 that

$$\gamma^1 = \int_0^1 d\gamma_1(t) = \gamma_1(T) - \gamma_1(0) = 1 - 0 = 1 \, . \;\; \gamma^2 = \int_0^1 d\gamma_2(t) = \gamma_2(T) - \gamma_2(0) = 0 - 0 = 0 \, .$$

Having these values, we compute via the shuffle product formula the quantities $\gamma^1\gamma^1$, $\gamma^2\gamma^2$, $\gamma^1\gamma^1\gamma^1$ and $\gamma^2\gamma^2\gamma^2$. Moreover, we obtain the values for γ^{11} , γ^{22} γ^{111} and γ^{222} . All these quantities hold with probability one, so the result hold also in expectation.
A useful observation for computing further terms in the expected signature is that the law of the SLE_κ in this setting is invariant under the conjugation mapping (this is a consequence of the fact that we are considering SLE from 0 to 1 in D) . Using the definition of the Young integral (the appropriate notion of integration for $p < 2$) we obtain that

$$\mathbb{E}[\gamma^{k_1 \dots k_n}] = (-1)^{|\{i | ki = 2\}|} \mathbb{E}[\bar{\gamma}^{k_1 \dots k_n}] \, .$$

This implies that any coordinate integral with an odd number of imaginary part components is automatically having zero expectation. In order to have the remaining terms in the three grading, we have to compute the remaining terms with an even number of imaginary components,i.e. γ^{122}, γ^{212}, and γ^{221}. By considering the shuffle products formula, we obtain that

$$0 = \gamma^2 \cdot \gamma^{12} = \gamma^{212} + 2\gamma^{122}\,,$$
$$0 = \gamma^2 \cdot \gamma^{21} = 2\gamma^{221} + \gamma^{212}\,.$$

In order to do so, we use a version of Green's Theorem for the Young integral. We consider the domain enclosed by the loop β that is the concatenation of the SLE_κ with the counter-clockwise arc from 1 to 0 on along the boundary of the disk. We denote this domain with $A(\gamma)$. By applying the shuffle product $\gamma^2\gamma^2$ on the curve up to time t that

$$\gamma^{221} = \int_0^1 \int_0^{t_1} \int_0^{t_2} d\gamma_2 d\gamma_2 d\gamma_1(t) = \frac{1}{2}\int_0^1 \gamma_2^2(t) d\gamma_1(t)\,.$$

Using Green's Theorem for the Young integral and by splitting the contour integral into the SLE path integral and the circular arc, we obtain that

$$\gamma^{221} = \frac{1}{2}\left(\frac{1}{8}\int_0^\pi sin^3(\theta)d\theta - 2\int_{A(\gamma)} ydxdy\right) = \frac{1}{12} - \int_{A(\gamma)} ydxdy\,.$$

Thus, by using the algebraic properties of the signature we arrived at the conclusion that all is needed to be understood is the quantity

$$\mathbb{E}\left[\int_{A(\gamma)} ydxdy\right] = \mathbb{E}\left[\mathbf{1}(x+iy \in A(\gamma))ydxdy\right] = \int_D y\mathbb{P}[x+iy \in A(\gamma)]dxdy\,. \tag{0.6.1}$$

We present the explicit computation of this quantity. Using the computation from the left-passage probability section, we denote the probability that SLE passes to the left of $r_0 e^{i\theta_0}$ by $\phi(\theta_0) := C_\kappa \int_0^{\theta_0} \sin^\lambda(t)dt$. Let $f(w) := \frac{w}{w+i}$, be the conformal map from $\mathbb{H}$ to D, and $g: D \to \mathbb{H}$ be its inverse. Using the change of variable formula and with $z = x+iy = f(w)$, we obtain that

$$\begin{aligned}\int_D yp(x,y)dxdy &= \int_D Im(z)\phi(arg(g(z)))dA(z)\\ &\int_{\mathbb{H}} Im(f(w))\phi(arg(w))|f'(w)|^2 dA(w)\end{aligned}$$

Changing to polar coordinates, we obtain that

$$\begin{aligned}&\int_{\mathbb{H}} Im(f(w))\phi(arg(w))|f'(w)|^2 dA(w)\\ &= \int_0^{\pi/2}(1-2\phi(\theta))\left(\frac{(2\sin^2(\theta)+1)(\pi/2-\theta-\sin(\theta)\cos(\theta))}{8\cos^4(\theta)} - \frac{\tan(\theta)}{4}\right)d\theta \quad = \int_0^{\pi/2}(1-2\phi(\theta))H(\theta)d\theta\,,\end{aligned}$$

where $H(\theta)$ is used to shorten the notation.
By inserting the definition of $\phi(\theta)$, and by applying Fubini's theorem, we obtain that

$$\begin{aligned}&\int_0^{\pi/2}(1-2\phi(\theta))H(\theta)d\theta\\ &\qquad = 2C_\kappa \int_0^{\pi/2} H(\theta)\int_0^{\pi/2} \sin^\lambda(t)dtd\theta\,,\\ &\qquad = 2C_\kappa \int_0^{\pi/2} \cos^\lambda(t)\int_t^{\pi/2} H(\pi/2-\theta)d\theta dt\,.\end{aligned}$$

Computing inner integral exactly

$$\int_t^{\pi/2} H(\pi/2-\theta)d\theta = \frac{1}{8}\left(1-\frac{\sin(t)-t\cos(t)}{\sin^3(t)}cos^\lambda(t\right)dt\,,$$

we obtain the final answer

$$\int_D yp(x,y)dxdy = \frac{1}{8}-\frac{C_\kappa}{4}\int_0^{\pi/2}\frac{\sin(t)-t\cos(t)}{\sin^3(t)}\cos^\lambda(t)dt\,.$$

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