Schramm-Loewner Evolutions (SLE) theory for beginners

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Contents

Introduction

The Loewner equation was introduced by Charles Loewner in 1923 in Complex Analysis and Geometric Function Theory and it played an important role in the proof of the Bieberbach Conjecture by Louis de Branges in 1985 . There are two versions of Loewner equation -radial and chordal- and they can be written as a partial differential equations or ordinary differential equations depending on the family of conformal maps that are studied. Moreover, the ordinary differential equation versions can be studied by considering a reverse time evolution also. In 2000 , Oded Schramm introduced a stochastic version of the Loewner equation in order to study the scaling limits of planar loop erased random walk and uniform spanning tree.

One motivation for studying the processes SLE_κ is their success in describing the scaling limits of various discrete models from planar Statistical Physics. For instance, it was proved that the scaling limit of loop erased random walk (with the loops erased in a chronological order) converges in the scaling limit to SLE_κ with $\kappa = 2$. Moreover, other two dimensional discrete models from Statistical Mechanics including Ising model cluster boundaries, Gaussian free field interfaces, percolation on the triangular lattice at critical probability, and Uniform spanning trees converge in the scaling limit to SLE_κ for values of $\kappa = 3$, $\kappa = 4$, $\kappa = 6$ and $\kappa = 8$ respectively. In fact, the use of Loewner equation along with the techniques of stochastic calculus, in this context gave a precise meaning to the passage to the scaling limit itself and proved rigorously the conformal invariance of the limits.

In addition, the introduction of the SLE gave new insights about some old problems. For instance, there were proved some long-standing open problems about planar Brownian motion such as the Mandelbrot Conjecture about the Hausdorff dimension of the Brownian frontier.

Loewner Differential Equation and SLE_κ

In this section, we study the interplay between the stochastic versions of chordal Loewner differential equations.

An important object in the study of the Loewner differential equation is the $\mathbb{H}$- compact hull that is a bounded set in $\mathbb{H}$ such that its complement in $\mathbb{H}$ is simply connected. To every compact $\mathbb{H}$-hull that we typically denote with K , we associate canonical conformal isomorphism $g_K \mathbb{H} \setminus K \to \mathbb{H}$ that is called the mapping out function of K. Note that the theory that we introduce it has at its core the conformal invariance structure. Thus, the general study of the objects of this theory can be mapped in one domain that is more convenient from the mathematical point of view. Most of the time, the choice is the upperhalf plane $\mathbb{H}$ with ∞ as a boundary point.

Given any compact $\mathbb{H}$-hull K , we construct the mapping out function $g_K : \mathbb{H}\setminus \to \mathbb{H}$. Using Riemann Mapping Theorem, we get uniqueness by imposing the *hydrodynamic normalization at* ∞ for g_t , i.e. we require that g_K looks like identity at ∞(i.e. it has no constant term and the complex derivative of it is 1 . The unique mapping $g_K(z) : H \to \mathbb{H}$, is constructed via Schwarz Reflection principle that is used to obtain a new mapping that has ∞ as an interior point, such that the map admits in the new domain a Laurent expansion there. Also, Schwarz reflection principle is used to prove that the new mapping maps the real line after a given value to the real line, so all the coefficients are real. Using the fact that the conformal automorphisms of $\mathbb{H}$ are of the form $f(z) = \sigma z + \mu$ for some positive σ and $\mu \in \mathbb{R}$, we obtain via suitable normalization and shifting, mapping at ∞ of the form

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2})\,, \quad |z| \to \infty\,.$$

The coefficient a_K that appears in the expansion at ∞ of the mapping has the traditional name *halfplane capacity* .

We start by introducing the Chordal Loewner Theory which establishes a one-to-one correspondence between continuous valued paths $(\zeta_t)_{t\geqslant 0}$ and an increasing families $(K_t)_{t\geqslant 0}$ of compact $\mathbb{H}$-hulls having a certain growth property.

Firstly, we define the conformal radius to be $rad(K) = \inf\{r \geqslant 0 : K \subset r\mathbb{D}+ \text{ x for some x} \in \mathbb{R}\}$. This is a very useful notion to estimate the distance between the points and its image for the mapping out function g_K , associated with the compact hull K .

Let $(K_t)_{t\geqslant 0}$ be a family of increasing $\mathbb{H}$-hulls, i.e. K_s is contained in K_t whenever $s < t$. For $K_{t+} = \cap_{s>t} K_s$ and for $s < t$, set $K_{s,t} = g_{K_s}(K_t \setminus K_s)$. We say that $(K_t)_{t\geqslant 0}$ has the *local growth property* if

$$rad(K_{t,t+h}) \to 0 \;\text{ as }\; h \to 0, \;\text{ uniformly on compacts in t}\,.$$

The first connection between the family of growing compact $\mathbb{H}$ -hulls and the real-valued path $(\zeta_t)_{t\geqslant 0}$ is done in the following Proposition.

Proposition 0.2.1. *Let $(K_t)_{t\geqslant 0}$ be an increasing family of compact $\mathbb{H}$-hulls having the local growth property. Then, $K_{t+} = K_t$ for all t . Moreover, the mapping $t \to hcap(K_t)$ is continuous and strictly increasing on $[0, \infty)$. Moreover, for all $t \geqslant 0$ there is a unique $\zeta_t \in \mathbb{R}$ such that $\zeta_t \in \bar{K}_{t,t+h}$, for all $h > 0$, and the process $(\zeta_t)_{t\geqslant 0}$ is continuous.*

The process $(\zeta_t)_{t\geqslant 0}$ is called the *Loewner transform* of $(K_t)_{t\geqslant 0}$.
The map $t \to hcap(K_t)/2$ is a homeomorphism on $[0,T)$ and by choosing τ to be the inverse of this homeomorphism, we obtain a new family of hulls K'_t in a new parametrization such that $\mathrm{hcap}(K'_t) = 2t$. This is the canonical parametrization that we use throughout the Thesis. We call this parametrization halfplane capacity.

In the following proposition is introduced the Loewner differential equation starting from the family of growing compact hulls. The main idea is that the local growth property of the hulls, gives a description in terms of a specific differential equation for the associated mapping out functions.

Proposition 0.2.2. *Let $(K_t)_{t\geqslant 0}$ be a family of increasing compact hulls in $\mathbb{H}$ satisfying the local growth property and that are parametrized by the halfplane capacity. Let $(\zeta)_t$ be its Loewner transform. Set $g_t = g_{K_t}$ and $\zeta(z) = \inf\{t \geqslant 0 : z \in K_t\}$. Then, for all $z \in \mathbb{H}$, the function $(g_t(z) : t \in [0, \zeta(z))$ is differentiable and satisfies the Loewner differential equation*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi_t}\,.$$

Moreover, if $\zeta(z) < \infty$ then $g_t(z) - \xi_t \to 0$ as $t \to \zeta(z)$.

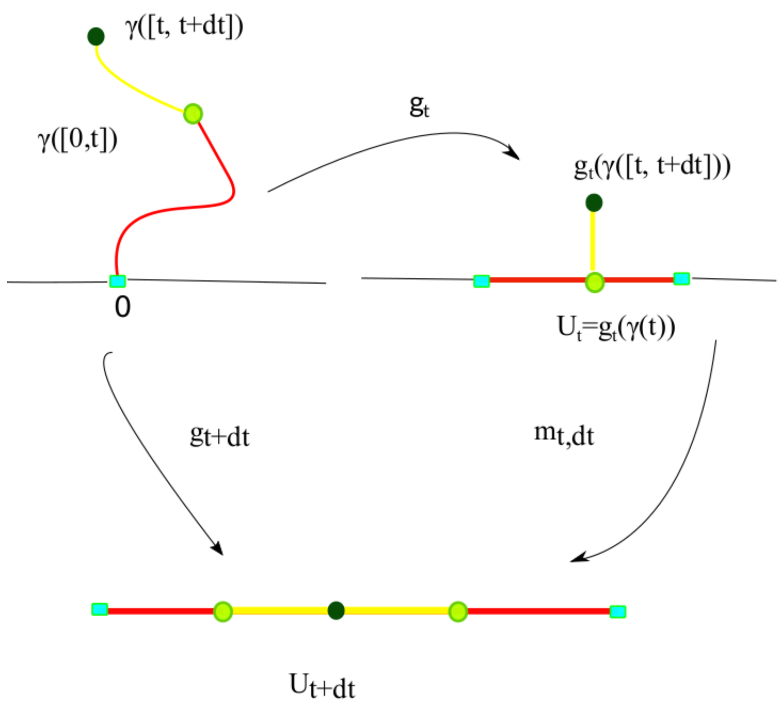


FIGURE 0.2.1. The Loewner differential equation in the case $K_t = \gamma[0,t]$.

The reverse situation is also true, i.e. from the driving function $(\zeta)_t$, we recover the family of growing compact $\mathbb{H}$-hulls .

Proposition 0.2.3. *For all $z \in \mathbb{C} \setminus \{\zeta_0\}$, there is a unique $\zeta(z) \in (0, \infty]$ and a unique continuous map $(g_t(z) : t \in [0, \zeta(z))$ in $\mathbb{C}$ such that, for all $t \in [0, \zeta(z))$ we have $g_t \neq \zeta_t$ and*

$$g_t(z) = z + \int_0^t \frac{2}{g_s(z) - \xi_s} ds \,,$$

and such that $|g_t(z) - \zeta_t| \to 0$ as $t \to \zeta(z)$, whenever $\zeta(z) < \infty$. Set $\zeta_0 = 0$ and define

$$C_t = \{z \in \mathbb{C} : \zeta(z) > t\} \,.$$

Then, for all $t \geqslant 0$ C_t is open and $g_t : C_t \to \mathbb{C}$ is holomorphic.
Moreover, the family of sets K_t= $z \in \mathbb{H} : \zeta(z) \leqslant t$ is an increasing family of compact $\mathbb{H}$-hulls having the local growth property. Moreover, hcap(K_t)=2t, and $g_{K_t} = g_t$, for allt. Moreover, the driving function ξ_t is the Loewner transform of $(K_t)_{t \geqslant 0}$.

The process $g_t(z) : t \in [0, \zeta(z)$ is called *the maximal solution to starting from* z and $\zeta(z)$ is called *lifetime of the solution*. Throughout the Thesis, we are mostly interested in the cases where the hull $K_t = \gamma([0,t])$, i.e. in the case when the hull is a curve.

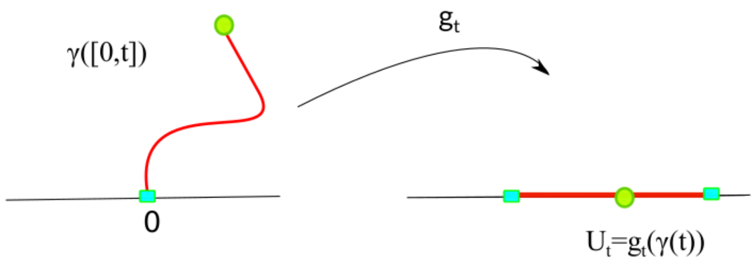


FIGURE 0.2.2. The conformal map that removes the curve grown up to time t

To gain some intuition about the Loewner equation and the impact of the driving function in the dynamics of the hulls, we briefly expose one example.

Example 0.2.4. In this example, we have $K_t = \gamma([0,t])$. For fixed $\alpha \in (0, \pi/2)$ we consider $\gamma(t) = r(t)e^{i\alpha}$ parametrized such that $hcap(K_t) = 2t$. Investigating the Loewner differential equation, by using the scaling map $z \to \lambda z$, we get that the mapping out functions $g_t = g_{K_t}$ satisfy $g_{\lambda^2 t}(z) = \lambda g_t(z/\lambda)$. Hence, for the driving function we have $\zeta_{\lambda^2 t} = \lambda \zeta_t$, so $\zeta_t = c_\alpha \sqrt{t}$. Fix τ such that $rad(K_\tau) = 1$ and note that, given $\varepsilon > -$ we can find $b > 1$, such that $g_\tau(b) \leqslant b + \varepsilon$ and $g_\tau(b-) \geqslant -b - \varepsilon$. Write $\delta-$ for the interval of ∂H_τ from $-b$ to γ_τ and $\delta+$ for γ_τ to b. Then, Brownian Motion starting from $y > 1$ is more likely to exit the more exposed part of the curve, i.e.

$$\mathbb{P}(B_{T(H_\tau)} \in \delta-) \geqslant \mathbb{P}(B_{T(H_\tau)} \in \delta+) \,,$$

and by multiplying with πy and letting $y \to \infty$, we get from Proposition A.2.1 that $g_\tau(\gamma_\tau) - g_\tau(-b) \geqslant g_\tau(\gamma_\tau) - g_\tau(b)$, Note that $g_\tau(\gamma_\tau) = \zeta_\tau = c_\alpha \sqrt{\tau}$ and so the previous inequality leads to

$2c_\alpha\sqrt{\tau} = 2\zeta_\tau \geqslant 2\varepsilon \geqslant 0$. . The equality case corresponds to the angle $\alpha = \pi/2$. If we exclude this case, we obtain a strict inequality. In fact c_α is decreasing with α such that $c_\alpha \to \infty$ as $\alpha \to 0$. Note the for greater c_α we have a greater angle of turn to the right . For a turn to the left , we take $\zeta_t = -c_\alpha\sqrt{t}$. So the driving function, is literally a 'driving' function, indicating that the tip of the curve turns right/left when the driver does.

Firstly, the (classical) chordal version of the Loewner partial differential equation produces a Loewner chain, i.e. a family of conformal maps from a domain of reference to a continuously decreasing family of simply connected domains. Formally, let $U(t)$ be a real valued function - the *driving term*, and let $f(t,z) : \mathbb{H} \to H_t$, where H_t is a continuously decreasing family of simply connected domains, be solutions to the chordal Loewner partial differential equation

$$\partial_t f(t,z) = -\partial_z f(t,z)\frac{2}{z - U(t)}\,, \quad f(0,z) = z, z \in \mathbb{H}\,. \tag{0.2.1}$$

Secondly, the (classical) chordal Loewner differential equation has two versions

(i) The forward chordal Loewner differential equation

$$\partial_t g(t,z) = \frac{2}{g(t,z) - U(t)}\,, \quad g(0,z) = z, z \in \mathbb{H}\,, \tag{0.2.2}$$

where the driving function U_t is a real valued function.

(ii) The backward chordal Loewner differential equation has the form

$$\partial_t h(t,z) = \frac{-2}{h(t,z) - U(t)}\,, \quad h(0,z) = z, z \in \mathbb{H}\,, \tag{0.2.3}$$

with $U(t)$ as before.

0.2.1. Candidates for the SLE driving function. In this subsection , we give a brief sketch of the proof that given the principles that our curves should satisfy, then the right candidate for the driver in Loewner equation is $\sqrt{\kappa}B_t$ where $\kappa \in \mathbb{R}$. When was defined by Oded Schramm, the purpose of it was to be the suitable candidate for a family of random curves in a domain $D \subset \mathbb{C}$ that were respecting two principles. The principles were motivated by the study of Schramm to give a precise meaning of the scaling limits of planar loop erased random walk with loops erased in the chronological order. When studying the planar loop erased random walk, Schramm realized that the scaling limit (if there exist any) should manifest some *Domain Markov Property* and *Conformal Invariance* , that are presented in the following.

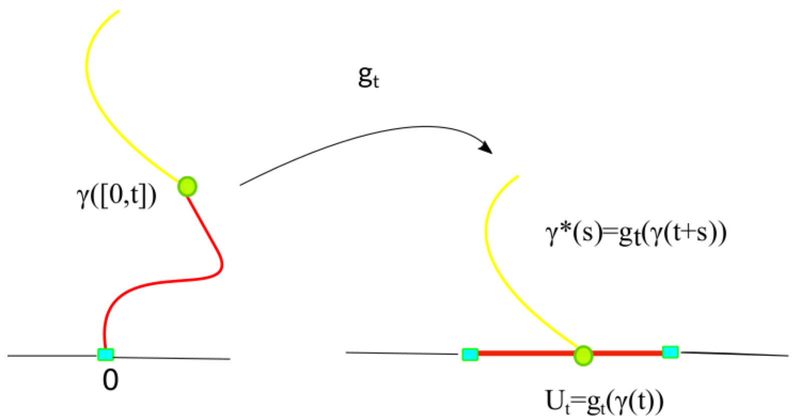


FIGURE 0.2.3. Conformal invariance of the curve

- [Domain Markov] Given the curve $\gamma[0,T]$, then the law of $\gamma[t,\infty]$ is a chordal SLE in $\mathbb{H}\setminus\gamma[0,T]$, from $\gamma(t)$ to ∞.
- [Conformal Invariance] The chordal SLE law in $\mathbb{H}\gamma[0,t]$ from $\gamma(t)$ to ∞ is the pull-back of the chordal SLE law in $\mathbb{H}$ from U_t to ∞.

If we require these two properties and a certain symmetry of the law on curves with respect to the imaginary axes, then the driving function in Loewner equation should be a $\sqrt{\kappa}B_t$. In the following, we give a sketch of the proof for this result of Oded Schramm that marks the introduction of SLE.

Sketch of the proof. Given the Domain Markov Property, the curve from U_t to ∞ that is denoted by $\gamma^*(s)$ has the law of a chordal SLE curve in $\mathbb{H}$ from U_t to $+\infty$ that is independent of $\gamma[0,t]$. Note that $\gamma^*(s)$ is determined by the driving function $U_{t+s}-U_t$, $s\geqslant 0$, and $\gamma[0,t]$ by the driving function U_s, $0\leqslant s\leqslant t$. We also have by the domain Markov property that $\gamma^*(s)$ is independent of $\gamma[0,t]$ means that the driving function after time t should be independent of the driving function up to time t, i.e. U_t is a Markov process.
Moreover, by conformal invariance (since g_t^{-1} is a conformal map, we have that $\gamma^*[0,t]$ is identical in law to $\gamma[0,t]$, so U_s ,$0\leqslant s\leqslant t$, should have the same law as $U_{t+s}-U_t$, $0\leqslant s\leqslant t$. So the increments of the U_t process have the same distribution and are independent, hence are i.i.d. Moreover, by writing $dU_t=bdt+\sigma dB_t$ then by the remarks that we obtain before about U_t, then a and b are forced to be constants. Thus, the conformal invariance and the Domain Markov property force the driving process U_t to be a Brownian motion with drift. If we use our last consideration, i.e. the law of the SLE curve should be invariant under reflections about the imaginary axis (i.e. U_t should be invariant under negations, we obtain that $b=0$ leaving $dU_t=\sigma dB_t$ for some constant $\sigma>0$ that we typically denote by $\sqrt{\kappa}$, due to reasons that will become obvious later.

□

SLE definition and first remarks

The Schramm-Loewner evolution (SLE) is a one-parameter (usually denoted by κ) family of random planar growth processes constructed as solution to Loewner equation when the driving term is a re-scaled Brownian motion. Thus, when studying the SLE_κ, in the upper half-plane, the corresponding families of conformal maps satisfy the equations (0.2.1), (0.2.2) and (0.2.3) in the formats :

(i) Partial differential equation version for the chordal SLE_κ in the upper half-plane

$$\partial_t f(t,z) = -\partial_z f(t,z) \frac{2}{z - \sqrt{\kappa} B_t}, \quad f(0,z) = z, z \in \mathbb{H}. \tag{0.3.1}$$

(ii) Forward differential equation version for chordal SLE_κ in the upper half-plane

$$\partial_t g(t,z) = \frac{2}{g(t,z) - \sqrt{\kappa} B_t}, \qquad g(0,z) = z, z \in \mathbb{H}, \tag{0.3.2}$$

(iii) Time reversal differential equation version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t,z) = \frac{-2}{h(t,z) - \sqrt{\kappa} B_t}, \qquad h(0,z) = z, z \in \mathbb{H}, \tag{0.3.3}$$

There are connections between these three formulations for studying families of conformal maps. The solution to the equation (0.3.1), i.e. the family of conformal maps satisfying (0.3.1) is related with the family of conformal maps satisfying (0.3.2) by the fact that at each instance of time t, the map $g_t(z)$ is the inverse of the map $f_t(z)$. In other words, the maps $f_t(z)$ "grow" the curve in the reference domain, while $g_t(z)$ maps conformally the slit domain obtained by the growing of the curve up to time t to the reference domain. The connection between the different versions of the Loewner equations that $g_{-t}(z)$ have the same distribution for all fixed t as the maps $f_t(z) - \zeta(t)$ that satisfy equation (0.3.3). This is proved in the next Lemma.

Lemma 0.3.1. *For all fixed $t \in \mathbb{R}$, the mappings $z \to g_{-t}(z)$ has the same distribution as the map $z \to f_t(z) - \zeta(t)$.*

Proof. Fix a time $s \in \mathbb{R}$, and let

$$\zeta_s(t) = \zeta(s+t) - \zeta(s).$$

By the shifting property of Brownian motion, we have that $\zeta_s(t)$ has the same distribution with ζ. Let us consider the mapping

$$\hat{g}_t(z) := g_{s+t} \circ g_s^{-1}(z + \zeta(s)) - \zeta(s),$$

Since g_t maps-down the curve up to time t and g_{t+s} maps down the curve obtain up to a later time $t+s$, we have that $\hat{g}_{-s}(z) = f_s(z) - \zeta(s)$. Since

$$\partial_t \hat{g}_t = \frac{2}{\hat{g}_t + \zeta_s - \zeta_{t+s}} = \frac{2}{\hat{g}_t + \zeta_s(t)},$$

the Lemma follows. □

0.3.1. SLE(0). In order to get some intuition, we first study the simplest version of chordal SLE_κ in the upper half-plane, i.e. the case $\kappa = 0$. Let us consider the process $(\gamma_t)_{t\geqslant 0} := 2it$ in $\bar{\mathbb{H}}$, the closed upper half-plane. The process is pictured in blue in Fig.2.1. By definition, this process evolves along the imaginary axis and the absolute value of it increases with the continuously increasing parameter t. This process belongs to the family of processes SLE_k for $k = 0$. Throughout this Transfer Thesis, we use the standard notations $K_t = \gamma(0,t] = \{\gamma_s : s \in (0,t]\}$, and $H_t = \mathbb{H} \setminus K_t$.

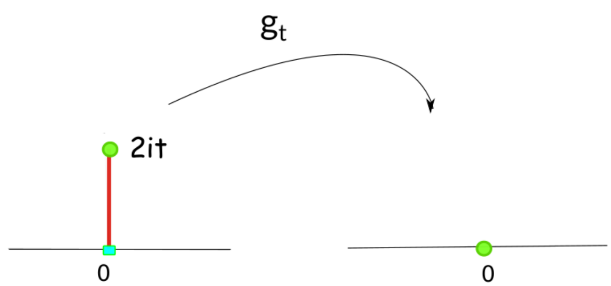


FIGURE 0.3.1. The conformal map that removes the slit grown up to time t

The mapping $g_t : H_t \to \mathbb{H}$ given by

$$g_t(z) = \sqrt{z^2 + 4t^2}$$

is a conformal isomorphism of the slit domain H_t and $\mathbb{H}$.
If the asymptotic behavior as $|z| \to \infty$ of $g_t(z)$ is given by

$$g_t(z) = z + \frac{2t}{z} + O(|z|^{-2}),$$

then we conclude using Riemann Mapping Theorem that this is the unique map with this property. Thus, we have a correspondence between the family of maps g_t and the path $(\gamma_t)_{t\geqslant 0}$. We now change the perspective and look at the evolution of points of the upper half-plane individually under the family of conformal mappings $(g_t)_{t\geqslant 0}$ via the identification $g_t(z) = z_t$.

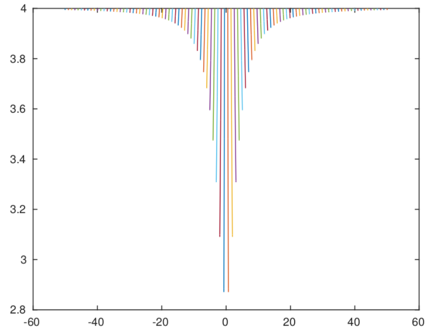
Using the fact that the family of maps $(g_t(z))_{t\geqslant 0}$ satisfies the Schramm-Loewner evolution for $\kappa = 0$, we obtain that

$$\frac{dz_t}{dt} = \frac{2}{\sqrt{z_t^2 + 4t^2}}.$$

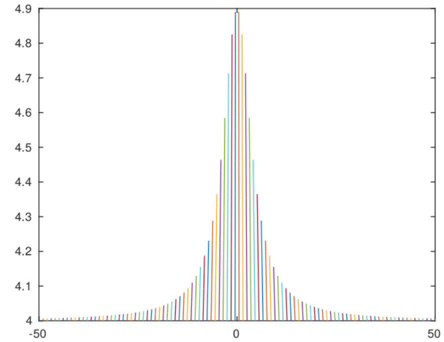
In fact, SLE_0 is obtained by iterating continuously a map $g_{\delta t}$ which "eats" infinitesimal bits of the form $(0, 2i\delta t]$ near the origin from the upper half plane.
We investigate separately the vector field $V(z)$ defined on $\mathbb{H} \setminus \{0\}$, given by

$$V(z) = \frac{2}{z} = \frac{2(x - iy)}{x^2 + y^2}.$$

(i) Our first observation is that $V(z_t) = \frac{2}{\sqrt{z_t^2 + 4t^2}}$.



(A) Forward flow before hitting time for $\kappa = 0$.



(B) Reverse flow for $\kappa = 0$.

(ii) Secondly, in order to not have singularities, we introduce the stopping time

$$\tau(z) = \inf\{t \geqslant 0 : \gamma_t = z\}\,.$$

By definition, we have that $\tau(z) = \frac{y}{2}$ if $z = iy$ and ∞ otherwise. We clearly have that if z is not purely imaginary, then $z_t \to 0$ as $t \to \tau(z)$. Thus, the maximal flow of the vector field $V(z_t)$ in $\mathbb{H} \setminus 0$ is given by $\big(g_t(z) : z \in \bar{\mathbb{H}} \setminus 0, t < \tau(z)\big)$.

(iii) A third observation is that the flow lines of $V(z_t)$ correspond to trajectories of points in which the imaginary part decreases as pictured in Fig.2.1. Thus, the flow lines $(z_t)_{t\geqslant 0} = (g_t(z))_{t\geqslant 0}$ of SLE_0 evolve to the left and right of the singularity at 0. Moreover, the path $(\gamma_t)_{t\geqslant 0}$ grows between the left and the right flow lines.

(iv) In the general case, for different values of k, we expect that if the singularity point (i.e. $\sqrt{\kappa}B_t$) moves infinitesimally to the left, then some of the left flow lines to be moved to the right, so the curve $(\gamma_t)_{t\geqslant 0}$ will turn to infinitesimally to the left. Moreover, the roughness of the Brownian driving term will be more apparent in the evolution of the path $(\gamma_t)_{t\geqslant 0}$ as pictured in Fig 2.2 for $\kappa = 1$ and $\kappa = 4.5$.

The simulations in MATLAB for the case $SLE(0)$ are describing the dynamics of a rectangle with long length and thin height in the upper half plane under if we run the forward or the reverse time flow of the Loewner equation. If we split the flow into the real and the imaginary part, we see that this respects the format of the equations(i.e. the imaginary part of the points is decreasing under the forward flow and the imaginary part is not changing the sign, also the points with bigger real value are moving more slowly under the flow than the other points).

One may start to ask simple questions about the flow in the case $\kappa = 0$. One may be interested in the time until you hit the origin under the forward flow. Is it clear that if z is purely imaginary, say $z = li$, then the time until you hit the origin is given by $l^2 i^2 + 4t = 0$ that gives $t = \frac{l^2}{4}$. When z is not purely imaginary, then under the forward flow it will never hit the origin or the real axis. We can prove that using the reverse flow. If z is purely real, then

(A) A sample of SLE_κ for $\kappa = 1$

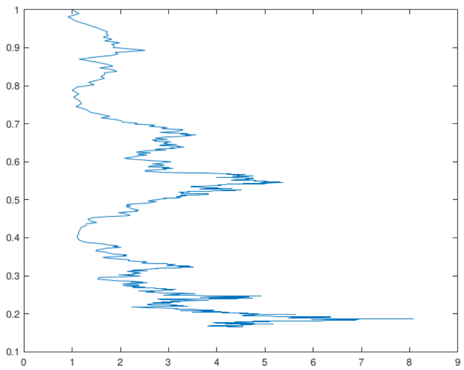
(B) A sample of SLE_κ for $\kappa = 4.5$

FIGURE 0.3.3. *Credits to Vincent Beffara*

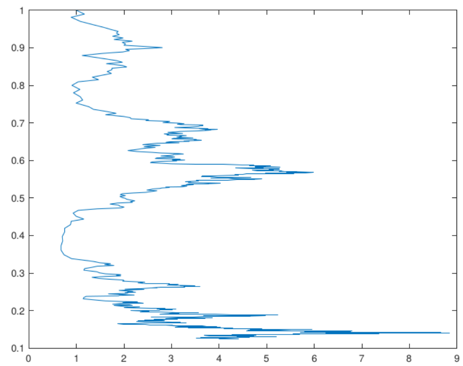
the equation $\dot{z}_t = \frac{2}{z_t}$ is an equation about real functions so if you start your dynamics on the real line, you remain on the real line for all the times under this flow. Now, let us assume that we start to evolve from a point in the upper half plane that is not on the $SLE(0)$ curve. Let us assume that under this dynamics we hit the real line after finite time. Then, running the reverse flow (i.e. the flow of the ordinary differential equation $\dot{z}_t = \frac{-2}{z_t}$, we do not come back to the original point of start because the flow (backward or forward) on the real axis remains real. The conclusion is that you do not hit the real line if you start from a point in the upper half-plane that is not on the curve.
Let us know investigate the dynamics of the points in the upper half-plane that are in the complement of the curve under the forward time Loewner flow. Splitting the ordinary differential equation in the real and the imaginary part, we obtain that

$$dX_t + idY_t = 2\frac{dt}{X_t + iY_t} = \left(\frac{2X_t}{X_t^2 + Y_t^2} - i\frac{2Y_t}{X_t^2 + Y_t^2}\right) dt.$$

Investigating the real and the imaginary part separately, we obtain the features of the dynamics. First, for we observe that the imaginary part of the points should decrease. Moreover, the decrease is faster , when the real part of the point is smaller. Thus, the points that are close to the curve (i.e. imaginary axis in this simple case) are more close to a "vertical fall" than the one that are having big real part. In addition, looking at the real part of the equation, we observe that the sign of it is not changed, i.e. is always increasing. Thus, fixing a w.lo.g positive real value and drawing a vertical line through that point, we can argue that the points on that line not go inside of the domain formed from curve real line from 0 up to the fixed real point and vertical line. The dynamics is moving the points to the left of that domain. Also, we observe that this dynamics becomes slower also with the increase of the real part (or absolute value of the point). This effect is apparent in the next pictures that consider the cases $(\kappa \neq 0)$.



(A) The flow for a point in the complement of the curve for SLE_4



(B) The flow for a point in the complement of the curve for SLE_6

The existence of the trace of SLE_κ

SLE scaling

The SLE process scales like its driving process B_t. This property will be used a lot in order to prove some other fundamental properties of the SLE process. We formalize this in the following Proposition.

Proposition 0.5.1. *Let us consider the g_t as being the solution to the chordal Loewner equation and let us take $r > 0$. Then $\hat{g}_t(z) := r^{-1} g_{r^2 t}(rz)$ has the same distribution with chordal SLE_κ, i.e. if γ is an SLE_κ path, then $\hat{\gamma}(t) := r^{-1}\gamma_{r^2 t}$ has the same distribution with γ.*

Proof. Clearly, we have that $\hat{g}_0(z) = z$, and the driving Brownian motion scales with the corresponding factors in time and space, so $\hat{B}_t := r^{-1} B_{r^2 t}$ has the same distribution as B_t. Using Loewner equation, we then have

$$\dot{\hat{g}}_t(z) = r\dot{g}_{r^2 t}(rz) = \frac{2r}{g_{r^2 t}(rz) - \sqrt{\kappa} B_{r^2 t}} = \frac{2}{\hat{g}_t(z) - \sqrt{\kappa}\hat{B}_t},$$

which concludes the proof. $\square$

Derivative expectation estimates

In order to provide the technical results for this part, we make a useful time change in the corresponding Loewner equation real and imaginary parts. With this new clock at hand, we obtain some technical Lemmas that are crucial in the proof of the existence of the trace for the SLE process. Recall the equations,

$$dX_t = \frac{-a2X_t}{X_t^2 + Y_t^2} dt - dW(t), \quad dY_t = \frac{aY_t}{X_t^2 + Y_t^2} dt. \tag{0.6.1}$$

We consider the time change $\sigma(t) = x_t^2 + y_t^2$, $t = \int_0^{\sigma(t)} \frac{ds}{x_s^2 + y_s^2}$. With the new time, we define the random variables $\tilde{Z}_t = Z_{\sigma(t)}$, $\tilde{X}_t = X_{\sigma(t)}$, and $\tilde{Y}_t = Y_{\sigma(t)}$.

We provide a martingale estimate for the backward Loewner differential equation. We start with the following proposition, in which the polynomial condition in the hypothesis comes from the fact that we are searching for martingales of the type $M_t := \tilde{Y}_t^\alpha (|\tilde{Z}_t|/\tilde{Y}_t)^\beta |h_t'(z_0)|^\gamma$, where α, β, γ depend on each other. This leads to the fact that they should satisfy a constraint.

Proposition 0.6.1. *Let r, b such that*

$$r^2 - (2a+1)r + ab = 0,$$

then

$$M_t := \tilde{Y}_t^{b-(r/a)} (|\tilde{Z}_t|/\tilde{Y}_t)^{2r} |h_t'(z_0)|^b,$$

is a martingale. Moreover,

$$\mathbb{P}(|h_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b} (|z_0|/y_0)^{2r} e^{t(r-ab)}.$$

Proof. By taking the complex derivative in the Loewner equation in the chain rule differentiation for the function $L_t = \log h_t'(z_0)$ we obtain that $L_t = -\int_0^t \frac{a}{Z_s^2} ds$, and in particular, $|h_t'(z_0)| = \exp\left(a \int_0^t \frac{\tilde{X}_s^2 - \tilde{Y}_s^2}{\tilde{X}_s{}^2 + \tilde{Y}_s{}^2} ds\right)$. Moreover, if we consider $\tilde{N}_t = \frac{\frac{\tilde{X}_t^2}{\tilde{Y}_t^2}}{1+\frac{\tilde{X}_t^2}{\tilde{Y}_t^2}}$, we obtain that

$$|h_t'(z_0)| = e^{-at} \exp\left(2a \int_0^t \tilde{N}_s ds\right).$$

In the $\sigma(t)$ time parametrization, looking at the equation for $\tilde{Y}_t$ we obtain a deterministic one $d\tilde{Y}_t = -a\tilde{Y}_t dt$, so in this time parametrization Y_t grows deterministic in an exponential manner $\tilde{Y}_t = Y_0 e^{at}$. At this moment, we can rephrase the formula for M_t as

$$M_t = y_0^{b-(r/a)} e^{-rt} (1 - \tilde{N}_t)^{-r} \exp(2ab \int_0^t \tilde{N}_s ds).$$

and by applying Ito's formula, we obtain that

$$dM_t = 2r\sqrt{\tilde{N}_t} M_t d\tilde{B}_t,$$

where $\tilde{B}_t = \int_0^{\sigma(t)} \frac{1}{\sqrt{X_t^2 + Y_t^2}} dB_t$ is the Brownian motion that we obtain in the time reparametrization. This shows that M_t is a martingale, hence

$$\mathbb{E}[M_t] = \mathbb{E}[M_0] = y_0^{b-(r/a)} (|z_0|/y_0)^{2r}.$$

Note that since for $r \geqslant 0$, $(|\tilde{Z}_t|/Y_t)^{2r} \geqslant 1$, then by Markov inequality, we have that

$$\mathbb{P}(|h_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b} (|z_0|/y_0)^{2r} e^{t(r-ab)}.$$

□

Corollary 0.6.2. *For every $0 \leqslant r \leqslant 2a+1$, there is a finite $c = c(a,r)$ such that for all $0 \leqslant t \leqslant 1$, $0 \leqslant y_0 \leqslant 1$, $e \leqslant \lambda \leqslant y_0^{-1}$, we have that*

$$\mathbb{P}(|h_t'(z_0)| \geqslant \lambda) \leqslant \lambda^{-b} (|z_0|/y_0)^{2r} \delta(y_0, \lambda),$$

where $b = [(2a+1)r - r^2]/a \geqslant 0$ and

$$\delta(y_0, \lambda) = \begin{cases} \lambda^{(r/a)-b}, & \text{if } r < ab, \\ -\log(\lambda y_0), & \text{if} r = ab, \\ y_0^{b-(r/a)}, & \text{if } r > ab. \end{cases}$$

Proof. From $dY_t = \frac{2Y_t}{X_t^2 + Y_t^2} dt$, we obtain that $dY_t \leqslant \frac{a}{Y_t}$, and hence $Y_t \leqslant \sqrt{2at + y_0^2} \leqslant \sqrt{2a+1}$. In the last inequality, we used that $t \leqslant 1$ and $y_0 \leqslant 1$. Using the exponential growth of Y_t in this time reparametrization, we obtain that $\tilde{Y}_t = \sqrt{2a+1}$ at time $T = \frac{\log\sqrt{2a+1} - \log y_0}{a}$. Therefore,

$$\mathbb{P}(|h_t'(z_0)| \geqslant \lambda) \leqslant \mathbb{P}(\sup_{0 \leqslant s \leqslant T} |h_s'(z_0)| \geqslant \lambda).$$

Using that $|h_t'(z_0)| = e^{-at} \exp\left(2a \int_0^t \tilde{N}_s ds\right)$ we obtain that

$|h'_{t+s}(z0)| \leqslant e^{as}|h'_t(z0)|$. So by addition of the probabilities, we have that

$$\mathbb{P}(\sup_{0\leqslant t\leqslant T}|h'_t(z0)| \geqslant e^a\lambda) \leqslant \sum_{j=0}^{[T]}\mathbb{P}(|h'_j(z_0)| \geqslant \lambda)\,.$$

Using the Schwarz-Pick Theorem for the upper halfplane we obtain that $h_t(z_0)| \leqslant \mathrm{Im}h'_t(z_0)/y_0 = e^{at}$. This gives a lower bound for the t that we are summing over and we obtain that via the Proposition 0.6.1 that

$$\begin{aligned}\mathbb{P}(\sup_{0\leqslant t\leqslant T}|h'_t(z0)| \geqslant e^a\lambda) &\leqslant \sum_{(1/a)\log\lambda\leqslant j\leqslant T}\mathbb{P}(|h'(z_0)| \geqslant \lambda)\\ &\leqslant \lambda^{-b}(|z_0|/y_0)^{2r}\sum_{(1/a)\log\lambda\leqslant j\leqslant T}e^{j(r-ab)}\\ &\leqslant c\lambda^{-b}(|z_0|/y_0)^{2r}\delta(y_0,\lambda)\,.\end{aligned}$$

□

Existence of the trace

Before stating the main Theorem of the section, we prove two propositions that together with 0.6.2 build the argument for the existence of trace of SLE_κ for $\kappa \neq 8$.

Proposition 0.7.1. *Suppose that g_t is a Loewner chain with driving function U_t and assume that there exist a sequence of positive numbers $r_j \to 0$ and a constant c such that*

$$|\hat{f}'_{k2^{-2j}}(2^{-j}i)| \leqslant 2^j r_j\,, k = 0, 1, \ldots, 2^{2j}-1\,,$$
$$|U_{t+s} - U_t| \leqslant c\sqrt{j}2^{-j}\,, 0 \leqslant t \leqslant 1, 0 \leqslant s \leqslant 2^{-2j}\,.$$

and

$$\lim_{j\to\infty}\sqrt{j}/\log r_j = 0\,.$$

Then $V(y,t) := \hat{f}_t(iy)$ is continuous on $[0,1]\times[0,1]$.

Proof. By differentiating $\partial_t f(t,z) = -\partial_z f(t,z)\frac{2}{z-U(t)}$, $f(0,z) = z, z \in \mathbb{H}$, we obtain that

$$\dot{f}'_t(z) = -f''_t(z)\frac{2}{z-U_t} + f'_t(z)\frac{2}{(z-U_t)^2}\,.$$

Bieberbach Theorem implies that $|f''_t(z)| \leqslant \frac{6|f'_t(z)|}{\mathrm{Im}(z)^2}$, and that $|f'_{t+s}(z)| \leqslant \exp\left[\frac{6s}{\mathrm{Im}(z)}\right]|f'_t(z)|$.

From hypothesis, we get that for $k = 0, 1, \ldots, 2^{2j}-1$

$$|f'_t(i2^{-j} + U_{k2^{-2j}})| \leqslant e^6 2^j r_j\,, k2^{-2j} \leqslant t \leqslant (k+1)2^{-2j}\,.$$

Using Distortion Theorem A.2.2, we get that for a univalent function on $\mathbb{D}$, we have that $|f'(z)| \leqslant 12|f'(0)|$ for $|z| \leqslant 1/2$. By iterating this, on a sequence of intersecting disks, we have that connect $z, w \in \mathbb{H}$ with $\mathrm{Im}(z), \mathrm{Im}(w) \geqslant y > 0$, for a conformal transformation $f : \mathbb{H} \to \mathbb{D}$, then we have that

$$|f'(w)| \leqslant 144^{(z-w)/y+1)} |f'(z)| .$$

In particular, by combining the hypothesis and $|f_t'(i2^{-j} + U_{k2^{-2j}})| \leqslant e^6 2^j r_j$ we obtain that there exist c and β such tat

$$|\hat{f}_t'(i2^{-j})| \leqslant e^{\sqrt{j}\beta} 2^j r_j , \;\; 0 \leqslant t \leqslant 1, \;\; j = 0, 1, 2, \ldots, 2^{-j} .$$

Using the distortion Theorem again but for a point that is not the lattice of space and time , we get

$$|\hat{f}_t'(iy)| \leqslant e^{\sqrt{j}\beta} 2^j r_j , \;\; 0 \leqslant t \leqslant 1, \;\; 2^{-j} < y < 2^{-j+1} , \;\; j = 0, 1, 2, \ldots, 2^{-j} .$$

By estimating the diameter of the derivative on this lattice, we get for $s \leqslant 2^{-2j}$ and $y, y_1 \leqslant 2^{-j}$ we get that $|\hat{f}_t(iy) - \hat{f}_{t+s}(iy)| \leqslant |\hat{f}_t(iy) - \hat{f}_t(i2^{-j})| + |\hat{f}_t(i2^{-j}) - \hat{f}_{t+s}(i2^{-j})| + |\hat{f}_{t+s}(i2^{-j}) - \hat{f}_{t+s}(iy)|$. The first and the third term are bounded by the estimate elaborated so far via

$$\| \hat{f}_t(iy) - \hat{f}_t(i2^{-j})| \leqslant \sum_{l=j}^{\infty} c e^{\beta \sqrt{l}} r_l .$$

From the assumption, the right hand side goes to 0 as $j \to \infty$. For the middle term, we have by using the estimate and the format of the partial differential equation that f solves, that

$$|\hat{f}_t(i2^{-j}) - \hat{f}_{t+s}(i2^{-j}) \leqslant 2s2^j \sup_{t \leqslant r \leqslant t+s} |f'(2^{-j})| \leqslant c r_j .$$

Since V is continuous already in $(0, \infty) \times [0, \infty)$ to establish the continuity on $[0, \infty) \times [0, \infty)$ it suffices to show that there exists a $\delta(\varepsilon)$ such that $\delta(0+) = 0$ and such that $|V(y,t) - V(y_1, s)| \leqslant \delta(y + y_1 + |t - s|)$, $0 \leqslant t, s, \leqslant t_0, y, y_1 > 0$. So by using the hypothesis, we conclude. $\square$

We need another result in order to conclude the existence of the trace for SLE process. For this we introduce the notion of *accesible point*. We call a point $z \in \hat{K}_t \setminus \cup_{s<t} \hat{K}_s$ $t-$ accessible if there exists a curve $\eta : [0,1] \to \mathbb{C}$, with $\eta(0) = z$ and $\eta(0,1] \subset H_t$.

Proposition 0.7.2. *Suppose g_T is a Loewner chain with driving function U_t and let $\hat{f}_t(z) = g_t^{-1}(z + U_t)$. Suppose that for each t, the limit $\gamma(t) = \lim_{y \to 0+} \hat{f}_t(iy)$, exists and the function $t \to \gamma(t)$ is continuous. Then g_t is the Loewner chain generated by γ.*

Proof. The proof relies on some Proposition 4.27 from Lawler [**?**] that shows together with the condition from the hypothesis that $\gamma(t)$ is the only $t-$ accesible point. Since $\gamma[0,t]$ is closed, the same Proposition 4.27 from [**?**] shows that $\partial H_t \cap \mathbb{H}$ is contained in $\gamma[0,t]$. $\square$

In order to prove this result, we need the Lemma from the introduction also.

Lemma 0.7.3. *For all fixed $t \in \mathbb{R}$, the mappings $z \to g_{-t}(z)$ has the same distribution as the map $z \to f_t(z) - \zeta(t)$.*

Theorem 0.7.4. *If $\kappa \neq 8$ the chordal SLE_κ is generated by a path with probability 1.*

Proof. By using the scaling of the SLE_κ, it suffices to prove the Theorem only for $t \in [0,1]$. According to the preliminary propositions it suffices to shoat that with probability 1 there exists an ε and a random constant c (because this estimate should hold for all $j's$ and $k's$) such that

$$|f'_{k2^{-2j}}(i2^{-j})| \leqslant c2^{j-\varepsilon}, j = 1, 2, \ldots, k = 0, 1, \ldots, 2^{2j},$$
$$|B_t - B_s| \leqslant c|t-s|^{1/2}|\log\sqrt{|t-s|}| \quad 0 \leqslant t \leqslant 1.$$

The second inequality is a consequence of the modulus of continuity for the Brownian motion (see A.1.1 in Appendix A). For the first inequality, we use a Borel-Cantelli Lemma along with 0.7.3 lemma to find c and ε such that for all $0 \leqslant t \leqslant 1$

$$\mathbb{P}(|h'_t(i2^{-j}| \geqslant 2^{j-\varepsilon}) \leqslant c2^{-(2+\varepsilon)j}.$$

Notice that we apply h'_t to points on the imaginary axis and that the corresponding $\lambda = 2^{j-\varepsilon}$.

If we consider $r = a + (1/4) < 2a + 1$ and $b = \frac{(1+2a)r-r^2}{a} = a + 1 + \frac{3}{16a}$, according to the Corollary 0.6.2. Thus, we are in the regime $r < ab$, so by the Corollary 0.6.2 we have that

$$\mathbb{P}(|h'_t(i2^{-j}| \geqslant 2^{j-\varepsilon}) \leqslant c2^{-j(2b-(r/a))(1-\varepsilon)}.$$

Investigating the exponent of 2, we obtain that $2b - (r/a) = 2a + 1 + 1/(8a) > 2$ provided that $a \neq 1/4$. So, we can apply Borel-Cantelli argument provided that $a \neq 1/4$ i.e. $\kappa \neq 8$ and finish the proof.

□

Bessel processes and the phases of SLE_κ

In this subsection, we analyze the phases of SLE_κ as a function of the parameter κ. We show that there exists two phase transition for values of $\kappa > 4$ and $\kappa \geqslant 8$. We determine these regimes not directly, but by studying the Loewner flow.

Consider the Loewner flow $(g_t(x) : t \in [0, \tau(x)), x \in \mathbb{R} \setminus 0$ on the real line associated with the Loewner differential equation when the driving function is $\sqrt{\kappa} B_t$ i.e. the flow associated with SLE_κ. Recall that $g_t(x) - \zeta_t \to 0$ as $t \to \tau(x)$, whenever $\tau(x) < \infty$. We introduce the notations

$$D = \frac{2}{\kappa}, \quad B_t = \frac{-\zeta(t)}{\sqrt{\kappa}}, \quad \zeta(x) = \tau(x\sqrt{\kappa}).$$

and set

$$X_t(x) = \frac{g_t(x\sqrt{\kappa}) - \zeta_t}{\sqrt{\kappa}},$$

for $t \in [0, \zeta(x))$.

With this notations, B_t becomes a standard Brownian motion that starts from 0 and we have that for $X_t(x) \neq 0$ for $t \in [0, \zeta(x))$ we have that

$$X_t(x) = x + B_t + \int_0^t \frac{D}{X_s(x)} ds.$$

We have two important properties of the Bessel process that are useful in our approach. The first one is that it has the following scaling property. For $\lambda \in [0, \infty)$ we consider the process

$$\tilde{X}_t(x) = \lambda X_{\lambda^{-2} t}(\lambda^{-1} x).$$

Note that the due the scaling property of the Brownian motion, we have that the family of processes $(\tilde{X}_t(x) : t \in [0, \tilde{\zeta}(x))$ and for each λ they are having the same distribution as $(X_t(x) : t \in [0, \tilde{\zeta}(x))$. Another very useful property is the so-called monotonicity property: for all $x < y$, we have that $\zeta(x) \leqslant \zeta(y)$ and $X_t(x) \leqslant X_t(y)$ for all $t < \zeta(x)$.

The Bessel processes manifest some phase transition that we capture in the following proposition. These results, give an interpretation of the Phase Transition of the SLE processes in terms of the understanding of the Loewner flow on the real axis as a Bessel process.

Proposition 0.8.1. *Let $x, y \in (0, \infty)$ with $x < y$. Then*

- *For $D \in (0, 1/4]$, we have*

$$\mathbb{P}(\zeta(x) < \zeta(y) < \infty) = 1.$$

- *For $D \in (1/4, 1/2)$, we have that*

$$\mathbb{P}(\zeta(x) < \infty) = 1.$$

$$\mathbb{P}(\zeta(x) < \zeta(y) < \infty) = \phi\left(\frac{y - x}{x}\right),$$

where ϕ is given by

$$\phi(\theta) \propto \int_0^\theta \frac{du}{u^{2-4a}(1-u)^{2a}}.$$

- *For $D \in [1/2, \infty)$, we have*

$$\mathbb{P}(\zeta(x) < \infty) = 0,$$

and moreover for $D \in [1/2, \infty)$, we have $X_t(x) \to \infty$ as $t \to \infty$ almost surely.

Proof. Fix $x > 0$ and write $X_t = X_t(x)$,and $\zeta = \zeta(x)$; For some fixed $r \geqslant 0$ we define the stopping time $T(r) = \inf\{t \in [0, \tau) : X_t = r\}$. We further assume that $0 < r < x < R$ for some real numbers r and R. We use the notation $S = T(r) \wedge T(R)$. From Proposition (insert cit) we have that $X_t \geqslant B_t + x$, for all $t \leqslant \tau$, so the hitting time of the R level is almost surely finite. For $a \neq 1/2$ define $M_t = X_t^{1-2a}$, for $t < \tau$.

Using Ito's formula for $M_t = X_t^{1-2a}$, we have that

$$dM_t = (1-2a)X_t^{-2a}dX_t - a(1-2a)X_t^{-2a-1}dt = (1-2a)X_t^{-2a}dB_t\,.$$

We thus obtain that M^S is a bounded martingale so that using optional stopping Theorem, we obtain that

$$x^{1-2a} = M_0 = \mathbb{E}(M_S) = r^{1-2a}\mathbb{P}(X_S = r) + R^{1-2a}\mathbb{P}(X_S = R)\,.$$

Considering $a \in (0, 1/2)$. by letting $r \to 0$ and by using that $\mathbb{P}(X_S = r) + \mathbb{P}(X_S = R) = 1$ we have that

$$\mathbb{P}(T(R) \leqslant \zeta) = (x/R)^{1-2a}\,. \tag{0.8.1}$$

Studying the asymptotics of the diffusion on the interval we obtain that when letting $r \to 0$ in the case $a \in (0, 1/2)$ we obtain that $\mathbb{P}(X_s = R) \to \mathbb{P}(T(R) \leqslant \zeta)$. Considering the other possibility , by taking $R \to \infty$ we obtain that $\mathbb{P}(X_s = r) \to \mathbb{P}(T(r) \leqslant \infty)$.

By letting $R \to \infty$ in 0.8.1 , we obtain that $\mathbb{P}(\zeta = \infty) = 0$.

Repeating the argument in the regime $a \in (1/2, \infty)$ we obtain that $\mathbb{P}(X_s = r) \to 0$, so $\mathbb{P}(T(R) \leqslant \zeta) = 1$, for all R, so $\mathbb{P}(\zeta = \infty) = 1$.

Using the fact that M solves an SDE with no drift, we obtain that M_t is a time-change of a Brownian motion. Using the fact that M_t must converge almost surely as $t \to \infty$ and that the quadratic variation of $[M]_\infty = (2a-1)^2 \int_0^\infty X_t^{-4a}$ must be finite, we obtain that $X_t \to \infty$ as $t \to \infty$.

For the special case $a = 1/2$, we take the process $M_t = \log X_t,$. With the same strategy as before, we arrive via Optional Stopping Theorem to the expression

$$\log x = \mathbb{P}(X_s = r)\log r + \mathbb{P}(X_s = R)\log R\,.$$

In the regime $a \in (1/2, \infty)$ we arrive at the same conclusion that $\mathbb{P}(\tau = \infty) = 1$. For $a \in (0, 1/2)$ we define $\xi(\theta) = \int_\theta^1 \frac{du}{u^{2-4a}(1-u)^{2a}}$. The mapping $\xi(\theta)$ comes in fact as a solution to the following ordinary differential equation

$$\xi^{''}(\theta) + 2\left(\frac{1-2a}{\theta} - \frac{a}{1-\theta}\right)\xi^{'}(\theta) = 0\,.$$

From the format of the denominator, the function also experiences a blow-up for $\theta = 0$ in the regime $a \in (1/4, 1/2)$, and remains finite for $a \in (1/4, 1/2)$. For fix $y \geqslant X$ we take $R_t = Y_t - X_t$ where $Y_t = X_t(y)$ and $\theta_t = R_t/Y_t$. By Ito's formula, we have that

$$dR_t = -\frac{aR_t dt}{X_t Y_t}\,, \quad d\theta_t = \left(\frac{\theta_t}{Y_t}\right)^2\left(\frac{1-2a}{\theta_t} - \frac{a}{1-\theta_t}\right)dt - \frac{\theta_t}{Y_t}dB_t\,.$$

If we take $N_t = \xi(\theta_t)$, then

$$dN_t = \xi'(\theta_t)d\theta_t + \frac{1}{2}\xi''(\theta_t)d\theta_t d\theta_t = -\frac{\xi'(\theta_t)\theta_t dB_t}{Y_t}\,.$$

We deduce that $(N_t : t \leqslant \zeta)$ is a local martingale. Using the fact that N_t is the time change of a Brownian motion, we obtain that N_t must converge to some limit as $t \to \zeta$. Using the fact that $\chi(\cdot)$ is a strictly decreasing function, we obtain that θ_t converges to some limit θ_ζ as $t \to \zeta$. Using the fact that X_t and Y_t become equal above the diagonal, we obtain that if $\zeta < \zeta(y)$ we have $\theta_\zeta = 1$ and so $N_\zeta = 0$. However, when $\zeta = \zeta(y)$ we have that $\theta_\zeta = 0$ almost surely. Note that, we necessarily have that $[N]_\zeta \leqslant \infty$ and

$$[N]_\zeta = \int_0^t \frac{\xi'(\theta_s)^2\theta_s^2}{Y_s^2}ds\,.$$

If $\theta_\zeta > 0$ then it follows that

$$\int_0^{\zeta(y)} \frac{1}{Y_s^2}ds < \infty\,.$$

Consider the random variables $A(x) = \int_0^\zeta \frac{1}{X_t^2}dt$, and the quantities $A_n(x) := \int_{T(2^{-n+1}x)}^{T(2^{-n}x)} \frac{1}{X_t^2}dt$, $n \geqslant 1$.

Using the strong Markov property for the Brownian motion, then the random variables $(A_n(x) : n \in \mathbb{N})$ are independent. Also, they have the same distribution by the scaling property. We can see that for $n = 1$ the integral $A_1(x) > 0$ almost surely. Then the whole series $\sum\limits_{n=1}^{\infty} A_n(x)$ is divergent and by using the fact that the series gives the value of the integral, we obtain that if $\zeta = \zeta_y$ then we have that $\theta_\zeta = 0$, (the integral is divergent for his particular time) .

In the case $a \in (0, 1/4], \zeta = \zeta_y$ would imply that $N_t = \xi(\theta_t) \to \infty$ as $t \to \zeta$, a contradiction with the fact that this is a bounded martingale, so $\mathbb{P}(\zeta < \zeta(y)) = 1$. On the other hand, in the regime $a \in (1/4, 1/2)$, the process N^τ is a bounded martingale so we can apply optional stopping and obtain that

$$\phi\left(\frac{y-x}{x}\right) = N_0 = \mathbb{E}(N_\tau) = \xi(0)\mathbb{P}(\zeta = \zeta_y) + 0\,.$$

Thus, we have shown that for $a \in (1/2, \infty)$ we have two regimes

$$\mathbb{P}(\zeta < \zeta(y)) = \begin{cases} 1, & \text{if } a \leqslant 1/4 \\ \phi(\frac{y-x}{x}), & a > 1/4. \end{cases}$$

□

These results translate from Bessel flow in terms of the path γ of an $SLE_(\kappa)$, in terms of hitting probabilities of the real line.

Proposition 0.8.2. *Let γ be an SLE_κ . Then we have the following behaviors in terms of κ .*

- *For $\kappa \in (0, 4]$, we have that $\gamma(0, \infty) \cap \mathbb{R} = 0$ almost surely.*

- *For $\kappa \in (4,8)$ and all $x, y \in (0,\infty)$, γ hits $[x,\infty)$ and*

$$\mathbb{P}(\gamma hits[x, x+y)) = \phi\left(\frac{y}{x+y}\right).$$

- *For $\kappa \in [8,\infty)$ then $\mathbb{R} \subseteq \gamma[0,\infty)$ almost surely.*

Proof. Fix $x, y \in (0, +\infty)$ and let $t > 0$. If $\gamma[0,t] \cap (x,\infty) = \emptyset$ then by the fact that the complement of a compact (hence closed set) is open, we have that there is a neighborhood of $[x,\infty)$ in $\mathbb{H}$ disjoint from $\gamma[0,t]$ which is then contained in H_t and $x \notin \bar{K}_t$. Thus we have that $\zeta(x) > t$. Also, if $\gamma_s \in [x,\infty)$ for some $s \in [0,t]$ then $\gamma_s \in \bar{K}_t$ and $\zeta(x) \leqslant \zeta(\gamma_s) \leqslant t$.Using the parameter $D = \frac{2}{\kappa}$ in the definition of the Bessel process, we obtain that for $\tau(x/\sqrt{\kappa})$- the lifetime of a Bessel process of parameter D -

$$\{\gamma \text{ hits } [x,\infty)\} = \{\tau(x/\sqrt{\kappa}) < \infty\}.$$

Using the Proposition 0.8.1, we obtain that

- If $D \in [1/2,\infty)$ then $\kappa \in (0,4]$, and $\mathbb{P}(\gamma \text{ hits } [x,\infty)) = 0$;

  Thus, $\mathbb{P}(\gamma \text{ hits } \mathbb{R} \setminus \{0\}) = \lim_{n\to\infty} \mathbb{P}(\gamma \text{ hits } (-\infty, 1/n] \cup [1/n, +\infty)) = 0$.

- If $D \in (1/4, 1/2)$, then $\kappa \in (4,8)$ so

$$\mathbb{P}(\gamma \text{ hits } [x,\infty))) = 1$$

$$\mathbb{P}(\gamma \text{ hits } [x, x+y))) = \phi\left(\frac{y}{x+y}\right).$$

- If $D \in (0, 1/4)$ then $\kappa \in [8,\infty)$ and so

$$\mathbb{P}(\gamma \text{ hits } [x, x+y))) = 1.$$

  So, almost surely for all rationals $x, y \in (0,\infty)$ we have that $\gamma_t \in [x, x+y)$ for some $t \geqslant 0$. Using that γ is continuous, we obtain that $[0,\infty) \subset \gamma(0,\infty)$ almost surely and by symmetry, we obtain that $\mathbb{R} \subset \gamma[0,\infty)$ almost surely.

□

Concerning the phases of SLE_κ, we prove the following results.

Theorem 0.8.3. *Let $(\gamma_t)_{t\geqslant 0}$ be a SLE. Then γ is almost surely transient for all κ's, i.e. $|\gamma_t| \to \infty$ as $t \to \infty$*

Theorem 0.8.4. *Let $(\gamma_t)_{t\geqslant 0}$ be a SLE. Then, we have the following behaviors when the parameter κ changes.*

- *For $\kappa \in [0,4]$, $(\gamma_t)_{t\geqslant 0}$ is a simple path almost surely.*

- *For $\kappa \in (4,8)$, $\bigcup_{t\geqslant 0} K_t = \mathbb{H}$, almost surely and for each $z \in \bar{H} \setminus \{0\}$, $(\gamma_t)_{t\geqslant 0}$ does not hit z almost surely.*

- *For $\kappa \in [8,\infty)$, $\gamma[0,\infty) = \bar{H}$, almost surely.*

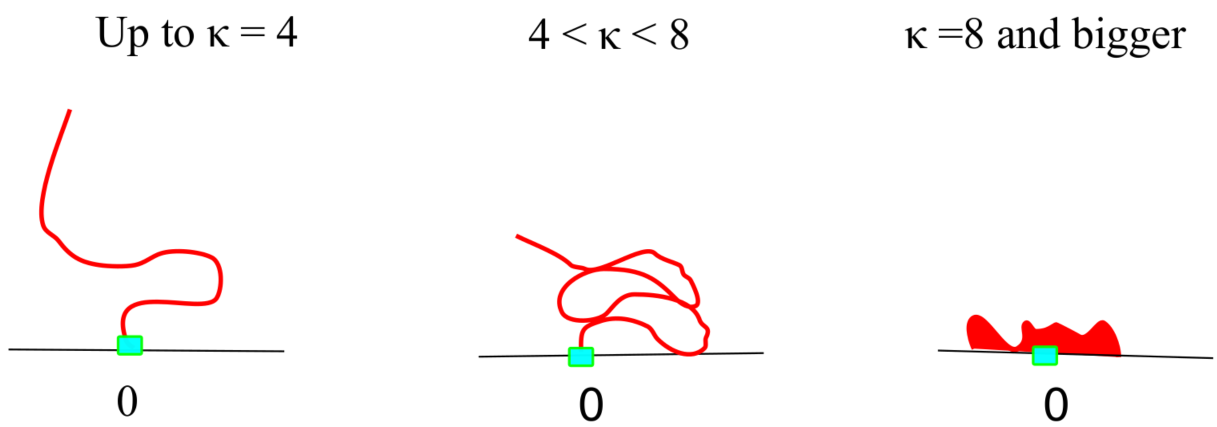


FIGURE 0.8.1. Possible Phases of SLE

The proofs of the Theorems is divided in several Lemmas.

Firstly, we study the regime $\kappa \in (0,4]$. Then, almost surely, $(\gamma_t)_{t\geqslant 0}$ is a simple curve and is transient. In order to prove this, we need the following Lemma:

Lemma 0.8.5. *Let $(\gamma_t)_{t\geqslant 0}$ be a simple path in $\mathbb{H}$ starting from 0. For fixed $r \in (0,1)$, define the first hitting time of the circle of radius r centered at 1, i.e.*

$$\tau = \inf\{t \geqslant 0 || \gamma_t - 1| = r\}.$$

Then,

$$|g_\tau(1) - \zeta_\tau| \leqslant r.$$

Proof. We consider the point $\gamma_\tau = a + ib$. We consider also the orthogonal line segments that connect $a+ib$ to 1, i.e. $I = (a, a+ib]$ and $J = [a \wedge 1, 1]$. We know that g_τ extends continuously to $\mathbb{R} \setminus \{0\}$ and to $g_\tau(\gamma_\tau) = \zeta_\tau$. By applying the conformal map g_τ the image of $I \cup J$ is a continuous path in $\bar{\mathbb{H}}$. that joins ζ_τ and $[g_\tau(1), \infty)$. So, by conformal invariance of Brownian motion, we have that

$$\begin{aligned}
\mathbb{P}(B_{T(\mathbb{H})} \in [\zeta_\tau, g_\tau(1)]) &\leqslant \mathbb{P}_{g_\tau(iy)}(B_{T(\mathbb{H}\setminus g_\tau(I))} \in g_\tau(I \cup J)) \\
&= \mathbb{P}_{iy}(B_{T(H_\tau \setminus I)} \in I \cup J) \\
&\leqslant \mathbb{P}_{iy}(\hat{B}_{T(\mathbb{H}\setminus I)} \in I^+ \cup J),
\end{aligned}$$

where I^+ denotes the right hand side of I. Note that $g_I(a+ib) = a$ and that $g_I(a+) = a+b$ and also that $g_I(1) = a+r$, where $a \leqslant 1$. We obtain that the Lebesgue measure on the interval of $I \cup J$ is bounded via $\mathrm{Leb}(g_I(I^+ \cup J)) \leqslant r$. Using the normalization at ∞ of g_τ, we have that $g_\tau(iy) - iy \to 0$ as $y \to \infty$. Then by the Proposition A.2.1 we obtain the desired estimate by a multiplication with πy and letting $y \to \infty$.

$\square$

Proposition 0.8.6. *Let $(\gamma_t)_{t\geqslant 0}$ be an SLE_κ with $\kappa \in (0,4]$. Then, almost surely, $(\gamma_t)_{t\geqslant 0}$ is a simple curve and is transient, i.e. $|\gamma_t| \to \infty$ as $t \to \infty$.*

Proof. Let $\kappa \in (0,4]$. We use the notation $K_{s,s+t} = g_s(K_{s+t} \setminus K_s)$ and $K_t^{(s)} = K_{s,s+t} - \zeta_s$. Then, $(K_t^{(s)})$ is an SLE_κ. Using the existence of the trace Theorem (Rohde-Schramm Theorem) we obtain, that almost surely, for all rational $s \geqslant 0$ and all $t \geqslant 0$, $g^{-1}_{K_{s,s+t}}$ extends continuously to $\mathbb{H}$, and $g^{-1}_{K_{s,s+t}}(z) \to \gamma_t^{(s)} + \zeta_s$ as $z \to \zeta_{s+t}$. Using again the existence of the trace Theorem with the composition of conformal maps, we obtain that

$$\gamma_{s+t} = \lim_{z\to\zeta_{s+t}, z\in\mathbb{H}} g_s^{-1}(g^{-1}_{K_{s,s+t}}(z)) = g_s^{-1}(\gamma_t^{(s)} + \zeta_s).$$

Since we have hcap(K_t)=$2t$ for all $t \geqslant 0$ almost surely, then there is no (non-degenerate) interval on which $\gamma(t)$ is constant. Thus, for $r, r' \geqslant 0$ with $r < r'$, there exists a rational $s \in \mathbb{Q}$ such that $\gamma_s \neq \gamma_r$. If we consider, $t = r' - s$, then we have that $\gamma_r \neq \gamma_{r'}$.

In order to prove the transience of the SLE_κ in this regime, we consider two cases.

- The case $\kappa \in (0,4)$. By the order of hitting the origin argument in 0.8.1 we know that $\inf_{t\geqslant 0}(g_t(1) - \zeta_t) > 0$ almost surely. Using the previous Lemma, we obtain that necessarily $\inf_{t\geqslant 0}|\gamma_t - 1| > 0$, almost surely. Let us set $t = 1$, i.e. we look at the moment $t = 1$ in time. We have that g_1 extends continuously to $\mathbb{R} \setminus \{0\}$ and that $\gamma_1 \in \mathbb{H}$. We consider the points $a^+ = \lim_{x\to 0+} g_1(x)$ and $a^- = \lim_{x\to 0-} g_1(x)$. When, mapping out the curve up to time 1, we obtain that $a^- < \zeta_1 = g_1(\gamma_1) < a^+$. We consider the sets

  $$B^+ = \{z \in H_1 | g_1(z) - a^+| < r^+\},$$

  $$B^+ = \{z \in H_1 | g_1(z) - a^+| < r^+\},$$

  where $r^\pm = \inf_{t\geqslant 0}|\gamma_t^1 + \zeta_1 - a\pm|$. Note that by scaling and $r^+ > 0$, almost surely. Note that, $\gamma_t \notin B^+ \cup \gamma(0,1] \cup B^-$ for all $t \geqslant 0$. By considering the simple paths $[0,1] \cup \gamma(0,1]$ and $\gamma(0,1] \cup [-1,0]$, we obtain that $B^+ \cup \gamma(0,1] \cup B^-$ is a neighborhood of the origin and hence $|\gamma_t|$ is almost surely positive and hence transient when $t \to \infty$. (i,e, if we consider $c = \lim_{t\to\infty}|\gamma_t|$, then by scaling $\mathbb{P}(c \geqslant r)$ is the same for all $r \geqslant 0$ so it is sufficient to prove that $\mathbb{P}(c > 0) = 1$ that we already did.)

- The case $\kappa = 4$. In this case, we develop the argument by looking at the dynamics under the conformal map g_t of three specific points by considering $X_t = g_t(1/4)$, $Y_t = g_t(1/2)$ and $Z_t = g_t(1)$. If it were the case that $|\gamma(t) - 1| \leqslant r$, for all $r \in (0,1)$, then it can be shown that the quantity $(Z_t - X_t)/(Y_t - X_t) \to \infty$. So, to show transience of the trace, we have to show that with probability 1 we have that $\lim_{t\to\infty} \frac{Z_t - X_t}{Y_t - X_t} < \infty$.

  So the chosen points are following a Bessel flow with parameter $a = 2/\kappa = 1/2$. By applying Ito's formula, we obtain that

  $$d\left[\log \frac{Z_t - X_t}{Z_t}\right] = -\frac{1}{2}\frac{1}{X_t Z_t}dt - \frac{1}{Z_t}dB_t\,, \tag{0.8.2}$$

  $$d\left[\log \frac{Z_t - X_t}{Y_t - X_t}\right] = \frac{1}{2}\frac{Z_t - Y_t}{X_t Y_t Z_t}dt\,.$$

  Hence we have the quantity $\lim_{t\to\infty} \log \frac{Z_t - X_t}{Y_t - X_t} = \log \frac{Z_0 - X_0}{Y_0 - X_0} + \frac{1}{2}\int_0^\infty \frac{Z_t - Y_t}{X_t Y_t Z_t}dt$.

First, note that by the choice of points, we have that $(Z_t - Y_t)/Z_t \leqslant (Z_t - X_t)/Z_t \leqslant R_t$ where R_t is defined by the following SDE

$$d[\log R_t] = -\frac{1}{2Z_t^2}dt - \frac{1}{Z_t}dB_t\,,$$

with initial condition $R_0 = (Z_0 - X_0)/Z_0 = 3/4$. We have this SDE for R_t because we want a quantity that stochastically dominates the quantity in 0.8.2, so it should solve a SDE with a higher drift coefficient for the same realization of Brownian motion. This SDE gives us the natural time change $\int_0^{r(t)} Z_s^2 ds = t$. Then by integrating the SDE obtain after using this time change, we obtain that $\log R_{r(t)} = \log(3/4) - t/2 - B_t$.In particular, with probability 1 there is a random constant $C_1 = C_1(\omega)$ such that $R_{r(t)} \leqslant C_1 e^{-t/4}$, for all t. Thus, for sufficiently large t we have that $R_t \leqslant 1/2$ i.e. $X_t \geqslant Z_t/2$. So we arrived to prove that with probability 1 we have that $\int_0^\infty R_t Z_t^{-2} dt$ is finite. So by a sequence of approximations and using the Strong Markov Property for Brownian Motion, we obtain that

$$\int_0^\infty R_t Z_t^{-2} dt = \sum_{k=0}^{\infty} \int_{r(k)}^{r(k+1)} R_t Z_t^{-2} dt \leqslant \sum_{k=0}^{\infty} C_1 e^{-k/4} < \infty\,,$$

that gives the desired conclusion.

□

In the regime $\kappa \in (4, 8)$ the curve is not simple or space-filling. We call this regime *swallowing phase*. In this regime, the curve has almost surely double-points and is transient. This is the content of the following result.

Proposition 0.8.7. *Let γ be a SLE_κ with $\kappa \in (4, 8)$. Then, with probability one,*

$$\bigcup_{t>0} \bar{K}_t = \bar{\mathbb{H}}\,,$$

and $\gamma[0,\infty) \cap \mathbb{H} \neq \mathbb{H}$. Also, $dist(0, \mathbb{H} \setminus K_t) \to \infty$, i.e. $|\gamma(t)| \to \infty$ as $t \to \infty$. Moreover, γ_t has double points, almost surely.

Proof. To show the first part of the proposition, we consider the following notion. We call$z \in \mathbb{H}$, *swallowed* if $T_z \leqslant \infty$ but $z \notin \cup_{t<T_z} K_t$. Note that, if z is swallowed then exists a ball B around it such that w is also swallowed for all $w \in B$. By Proposition 0.8.1, we have that there exists $x > 1$ such that $T_x = T_1$. By scaling of the Bessel process, we can obtain that such a point $x > 1$ exists almost surely. Also, the point with this hitting time of the origin and largest absolute value is $\gamma(T_1)$. If we consider $\varepsilon = dist(1, \gamma[0, T_1])$, then all points in $z \in \mathbb{H} \cap B(1, \varepsilon)$ are swallowed. This shows that $\gamma[0,\infty) \cap \mathbb{H} \neq \mathbb{H}$. (i.e. on the event $\{\gamma_{\zeta x} < y\}$, there is a neighborhood of x in $\mathbb{H}$ which does not meet γ and from this positive probability we obtain via Blumenthal's 0-1 law an almost sure result). If we consider T to be the first time that -1 and 1 are swallowed, then by topological consideration there is a disk D about origin such that $D \cap \mathbb{H} \subset K_T$.

In particular, for each $u > 0$ there is an $\varepsilon > 0$ such that $\mathbb{P}(B(0,\varepsilon) \cap \mathbb{H} \subset K_T) \geqslant 1 - 2u$. Moreover, there exists a $t = t_{\varepsilon,u}$ such that $\mathbb{P}(B(0,\varepsilon) \cap \mathbb{H} \subset K_t) \geqslant 1 - u$. So by scaling, this inequality holds for all ε (for a t depending on ε). This gives the first assertion and also the transience via the following zero-one argument. The fact that $\mathbb{P}(B(0,\varepsilon) \cap \mathbb{H} \subset K_t) \geqslant 1 - u$

gives that $\mathbb{P}(dist(0, \mathbb{H} \setminus K_t) = \delta > 0$ for some $t > 0$. This extends to all t by scaling with the same δ. So, the distance between 0 and K_t is strictly greater than 0 for all t with probability δ. $\mathbb{P}(dist(0, H_t) > 0$ for all t¿0) $= \delta$ and via Blumenthal's $0 - 1$ law, $\delta = 1$. Finally, via a scaling argument, we obtain that for all $r < \infty$ and $t \to \infty$

$$\mathbb{P}(\mathrm{dist}(0, H_t) \leqslant r) = \mathbb{P}(\mathrm{dist}(0, H_1) \leqslant r/\sqrt{t}) \to 0\,.$$

In order to show that γ_t has double points almost surely, we consider the set non-decreasing set in t, $A_t := \{\gamma_s = \gamma'_s \text{for some distinct s, s'} \in [0, t]\}$. By scaling we argue that the sets A_t have the same probability p. We obtain then that $p = \mathbb{P}(\cap_t A_t)$ and as $\cap_t A_t \in \mathcal{F}_{0+}$, we obtain that via Blumenthal's 0-1 law that $p = 0$ or $p = 1$. Furthermore, by topological considerations we have that $\partial K_1 \cap \mathbb{H} \subset \gamma[0, 1]$, so γ has double points (because it filled somehow the hull) with positive probability, and by the previous result almost surely.

□

Remark 0.8.8. The regime $\kappa > 8$ is the regime when the curve γ_t is space-filling. This result is proved in detail in Thereom 7.9 in [**?**].

The Hölder continuity of SLE_κ maps

With the typical notations from the Bessel process approach to the Loewner flow, we define $\alpha(\kappa) = \alpha(2/a)$ to be the supremum over all α such that if $t > 0$ a.s. the function $z \to g_t^{-1}(z)$ is Hölder α-continuous. Using the scaling of SLE_κ this notion is independent of the choice of t and by definition g_t^{-1} is Hölder α-continuous if and only if $\hat{f}_t$ is Hölder α-continuous. We have the following asymptotics statement for $\alpha(\kappa)$.

Proposition 0.9.1. *For every $\kappa \neq 4$, $\alpha_0(\kappa) > 0$. Moreover,*

$$\lim_{\kappa \to \infty} \alpha_0(\kappa) = 1\,, \quad \lim_{\kappa \to 0+} \alpha_0(\kappa) = \frac{1}{2}\,.$$

Proof. Using the scaling of the SLE we can restrict our analysis to $t \leqslant 1$ and to the rectangle $-1 < \mathrm{Re}(z), \mathrm{Re}(w) < 1$, and also to $0 < \mathrm{Im}(z), \mathrm{Im}(w) \leqslant 1$. It suffices to find a c such that

$$|\hat{f}_t(\frac{j}{2^n} + \frac{i}{2^n})| \leqslant c2^n 2^{-\alpha n}\,, \;\; n = 0, 1, 2, \ldots,\,, \;\; j = 2^{-n}, \ldots 2^n\,,$$

because by the Distortion Theorem A.2.2 applied for a point in the grid and for one point with coordinates situated on the complement of the grid in the rectangle, we would have that $|\hat{f}_t{}'(x + iy)| \leqslant c_1 \mathrm{Im}(y)^{\alpha - 1}$, for $-1 \leqslant x \leqslant 1\,, 0 < y \leqslant 1$. By integrating this formula and using the fact that the absolute value of the integral is less than the integral of the absolute value, we obtain the conclusion.

Using Corollary 0.6.2 we obtain that for any $0 < r \leqslant 2a + 1$ and $b = \frac{(1+2a)r - r^2}{a} \geqslant 0$ we have that

$$\mathbb{P}(|\hat{f}_t(\frac{j}{2^n} + \frac{i}{2^n})| \leqslant c2^n 2^{-\alpha n}) \leqslant c(1 + 2^{2n})^r 2^{-n(2b - (r/a))(1-\alpha)}\,, \;\; \text{if } r < ab\,,$$
$$\mathbb{P}(|\hat{f}_t(\frac{j}{2^n} + \frac{i}{2^n})| \leqslant c2^n 2^{-\alpha n}) \leqslant c(1 + 2^{2n})^r 2^{-nb(1-\alpha)} n\,, \;\; \text{if } r = ab\,,$$
$$\mathbb{P}(|\hat{f}_t(\frac{j}{2^n} + \frac{i}{2^n})| \leqslant c2^n 2^{-\alpha n}) \leqslant c(1 + 2^{2n})^r 2^{-nb(1-\alpha)} 2^{n((r/a) - b)}\,, \;\; \text{if } r > ab\,,$$

By choosing according the parameter α we obtain that if

$$\alpha < \frac{2b - (r/a) - 2r - 1}{b + \max(0, b - (r/a)} ,$$

then

$$\sum_{n=1}^{\infty} \sum_{j=2^{-n}}^{2^n} \mathbb{P}(|\hat{f}_t(\frac{j}{2^n} + \frac{i}{2^n})| \leqslant c 2^n 2^{-\alpha n}) < c \sum_{n=1}^{\infty} 2^{-n\varepsilon} < \infty ,$$

for some c and $\varepsilon > 0$. And applying the Borel-Cantelli Lemma, we obtain that

$$|\hat{f}_t(\frac{j}{2^n} + \frac{i}{2^n})| \leqslant c 2^n 2^{-\alpha n} , \quad n = 0, 1, 2, \ldots, , \quad j = 2^{-n}, \ldots 2^n ,$$

Following a careful analysis of the bound $\alpha < \frac{2b-(r/a)-2r-1}{b+\max(0,b-(r/a)}$ we have that for $a < 1/4$ we can choose $r = 2a$ and b becomes 2. This choice gives $2b - r/a - 2r - 1 > 0$. On the other hand , if $a \geqslant 1/6$ we let $r = (a/2) + (1/4)$ and then $b = 3a/4 + 3/4 + 2/(16a)$. This choice makes $2b - r/a - 2r - 1$ positive unless $a = 1/2$ and here is where we loose the case $\kappa = 4$. So the denominator is always positive unless $a = 1/2$. If we let $r = \sqrt{a}$ and $b = 2\sqrt{a} - 1 + a^{-1/2}$, we see that the paths are α-Hölder continuous for all

$$\alpha < \frac{2\sqrt{a} - 3 + a^{-1/2}}{2\sqrt{a} - 1 + a^{-1/2} + \max(0, 2\sqrt{a} - 1)} ,$$

and by taking the limits in κ (i.e. in $2/a$) we obtain the asymptotics for the Hölder exponent. □

An important observable: the left passage probability of chordal SLE_κ in $\mathbb{H}$

In this section we give an overview of Professor's Oded Schramm paper *A percolation formula* in which is computed the probability that chordal SLE passes through the left or to the right of some fixed point z_0. The terminology used in the SLE literature for this kind of quantities is *observable*. This explicit formula was used to prove other results involving SLE and is used also in the paper of Prof. Brent Werness that we will present in detail in the last Chapter of this essay. We start with mentioning the relevant properties of SLE that are used in order to describe this *observable*.

Some of the properties of the trace are that γ is a.s. transient , i.e. $\lim_{t\to\infty} |\gamma(t)| = \infty$ and that when $\kappa \in (0, 8)$ it almost surely does not hit fixed points in the upper half plane, i.e. for every $z \in \mathbb{H}$, we have $\mathbb{P}[z \in \gamma[0, \infty)] = 0$. It is then natural to ask whether if the SLE trace passes to the left or to the right of a fixed point $z_0 = x_0 + iy_0$ in the upper half-plane. The quantity that describes this phenomena is the winding number of the following curve. Let $\mathbb{D}$ be the unit disk. We define β_t to be the curve that connects the tip γ_t to 0 via β_t is the path that follows the arc $|\gamma_t|\partial\mathbb{D}$ from the tip to the real number $\gamma(t)$ clockwise and then the real axis up to 0, i.e. is the circular arc that connects the tip with the point $|\gamma(t)|$ situated at the real axis concatenated with the segment $[0, |\gamma(t)|]$. From the topological point of view γ passes to the left of the z_0 if the winding number of $\gamma[0, t] \cup \beta_t$ around z_0 is 1 for all large t. This leads to the following Theorem.

Theorem 0.10.1. *Let $\kappa \in [0,8)$ and let $z_0 = x_0 + iy_0 \in \mathbb{H}$. Then the trace γ of chordal SLE_κ satisfies*

$$\mathbb{P}[\gamma\, passes\, to\, the\, left\, of\, z_0] = \frac{1}{2} + \frac{\Gamma(4/\kappa)}{\sqrt{\pi}\Gamma(\frac{8-\kappa}{2\kappa})}\frac{x_0}{y_0}F_{2,1}(\frac{1}{2},\frac{4}{\kappa},\frac{3}{2},\frac{-x_0^2}{y_0^2})\,.$$

In order to prove this result , we need the following Lemma. Let $x_t := \mathrm{Re} g_t(z_0) - W(t)$, $y_t := \mathrm{Im} g_t(z_0)$ and $w_t = \frac{x_t}{y_t}$.

Lemma 0.10.2. *Almost surely, γ is to the left (respectively to the right) of z_0 if $\lim_{t\to\tau(z_0)} w_t = +\infty$. (respectively, $-\infty$.)*

Proof. [Sketch of the proof of the Lemma] The proof is divided into two sections. First, let us suppose that $\kappa \in [0,4]$. In this regime we know from [**?**] that a.s. γ is a simple path and moreover by the previous remark about hitting of the points in the complex plane by the curve that $\tau(z_0) = \infty$. We consider the arc of radius r for some $r >> |z_0|$ (r much more larger than $|z_0|$) and we consider τ_r to be the first time that the tip of the curve touches the arc of radius r, i.e. $|\gamma(t)| = r$. We further define the domain $D_+ \subset \mathbb{H}$ be the domain whose boundary is composed of $[0,r]\cup\gamma[0,\tau_r]$ and an arc on the boundary of $r\partial\mathbb{D}$. Furthermore $D_-(r) = r\mathbb{D}\backslash D_+(r)$. Given γ we start a Brownian motion from z_0. For very large r the Brownian motion will hit with very high probability $\gamma[0,\tau_r]\cup\mathbb{R}$ before exiting the disk $r\partial\mathbb{D}$. If we consider $z_0 \in D_+(r)$ then the Brownian motion started from z_0 is likely to hit the SLE trace or the interval $[0,r]$ from within $D_+(r)$. If we map-down the curve up to time τ_r, (i.e. by definition this is what the mapping g_{τ_r} is doing) then the tip it is mapped to $W(\tau_r)$. We further use that the harmonic measure is conformally invariant. Thus, the harmonic measure in $\mathbb{H}$ of $[W(\tau_r,\infty)]$ from $g_{\tau_r}(z_0)$ (this is where the point z_0 is mapped into the new domain) is close to 1 if $z_0 \in D_+(r)$ and close to 0 if $z_0 \in D_-(r)$. Using the definition of x_t and y_t we have that the harmonic measure in $\mathbb{H}$ of $[0,\infty)$ from $x_{\tau_r} + iy_{\tau_r}$ is close to 1 if $z_0 \in D_+(r)$ and close to 0 if it is in the complement. By definition of w_t if follows that w_t is either far left or either far right (this is the information that we get from the asymptotics of harmonic measure), i.e. w_{τ_r} is close to $+\infty$ or to $-\infty$. This finishes the argument in the case $\kappa \in [0,4]$. For $\kappa \in (4,8)$ the analysis is similar except that we are dealing with a more complicated picture. In this regime γ is not a simple path and $\tau(z_0)$ is finite (z_0 is in a bounded component of $\mathbb{R}\cup\gamma[0,\tau(z_0)]$) according to [**?**]. Clearly, z_0 is not in a bounded component of $\mathbb{R}\cup\gamma[0,t]$ when $t < \tau(z_0)$. Thus, at time $\tau(z_0)$ the path γ encloses a loop around z_0. To answer the question of left/right passage the question becomes equivalent to z_0 being surrounded by a clockwise loop or a counter-clockwise loop . By applying the conformal mapping that maps down the hull as above, we obtain as in the simple curve case the asymptotics of $w_t \to +\infty$ or $-\infty$, as $t \to \tau(z_0)$. □

We now give a brief proof of the Theorem.

Proof. [Sketch of the proof of the Theorem]

First, we split the Loewner differential equation into the equation for the real part and the equation for the imaginary part. This gives

$$dx_t = \frac{2x_t}{x_t^2 + y_t^2}dt - dW(t)\,, \quad dy_t = \frac{-2y_t}{x_t^2 + y_t^2}dt\,. \tag{0.10.1}$$

Applying Ito's formula for the function $f(x,y) = \frac{x}{y}$ and using the fact that the quadratic variation terms all vanish due to the fact that $f''(x) = 0$, we obtain that

$$dw_t = -\frac{dW_t}{y_t} + \frac{4w_t}{x_t^2 + y_t^2} dt\,.$$

We naturally provide the time change

$$u(t) = \int_0^t \frac{dt}{y_t^2}\,,$$

and setting

$$\tilde{W}(t) = \int_0^t \frac{dW(t)}{y_t}\,,$$

we obtain a new SDE

$$dw = -d\tilde{W} + \frac{4w}{w^2+1} du\,. \tag{0.10.2}$$

Note that we choose this time change in order to have $\frac{\tilde{W}}{\sqrt{\kappa}}$ a Brownian motion as a function of u (it can be easily checked via the fact that the quadratic variation of the $\tilde{W}$ process that is defined via an SDE with no drift is u). Also in this new SDE we are left with one single variable that is the argument of the point. This is a consequence of the scale invariance. We apply right now classical Stochastic Analysis results by considering the diffusion for w. Given a starting point $\hat{w}$ given $a, b \in \mathbb{R}$with $a < \hat{w} < b$, we compute the probability that w will hit b before a. We denote this probability with $h_{a,b}(\hat{w})$. From standard Stochastic Analysis we know that $h(w_u)$ is a local martingale. We are now trying to obtain the explicit form of this martingale. Considering (for the moment) h smooth and applying Ito's formula, we obtain that h should satisfy the following ODE (this ODE is the result of considering the coefficient in front of dt to be 0 since h is a martingale so there is no drift term)

$$\frac{\kappa}{2} h^{"}(w) + \frac{4w}{w^2+1} h^{'}(w) = 0\,,\ \ h(a) = 0\,,\ h(b) = 1\,. \tag{0.10.3}$$

By the maximum principle , these equations have a unique solution and moreover these solutions can be expressed in terms of hypergeometric functions like

$$h(w) = \frac{f(w) - f(a)}{f(b) - f(a)}\,, \tag{0.10.4}$$

where $f(w) := F_{2,1}(\frac{1}{2}, \frac{4}{\kappa}, \frac{3}{2}, -w^2)w$. Using this explicit form of h we can dispose the assumption that h is smooth because we have automatically that for h of this specific form the dt part is automatically 0 since it solves the corresponding ODE. So in some sense, we guessed the martingale. By uniqueness it follows easily that it must be also equal to h. By standard theory about hypergeometric functions for $\kappa < 8$ it follows that

$$\lim_{w \to \infty} f(w) = \pm \frac{\sqrt{\pi}\Gamma((8-\kappa)/(2\kappa)}{2\Gamma(4/\kappa)}\,. \tag{0.10.5}$$

In particular, the limit is finite, which shows that $lim_{b \to \infty} h_{a,b}(w) > 0$ for all $w > a$, hence the diffusion is transient (i.e. there is a non-zero probability to hit the "infinity" level). We have the formula

$$\mathbb{P}[\lim_{u \to +\infty} w_u = +\infty] = \frac{f(\hat{w}) - f(-\infty)}{f(\infty) - f(-\infty)}\,. \tag{0.10.6}$$

The previous Lemma gives the desired conclusion.

$\square$

Appendix A

Pathwise estimate for Brownian Motion

Proposition A.1.1. *Let B_t be a standard one-dimensional Brownian motion. Let $M_n = \max_{j=0,\dots,2^n-1} \sup_{0\leqslant t\leqslant 2^{-n}} |$ $B_{j2^{-n}}|$. Then*

$$\lim_{n\to\infty} \frac{M_n}{\sqrt{2^{-n}\log(2^n)}} = \sqrt{2}\,.$$

Proof. For X a standard normally distributed random variable we obtain via direct estimates of the density the following bounds

$$e^{-\frac{(a+1)^2}{2}} \leqslant \sqrt{2\pi}\mathbb{P}(X \geqslant a) \leqslant e^{-\frac{a^2}{2}}\,.$$

Using the independence of increments and the scaling property of Brownian motion, we have for $c > 0$ that

$$\begin{aligned}
\mathbb{P}(M_n \leqslant c2^{-\frac{n}{2}}\sqrt{n}) &\leqslant \prod_{j=0}^{2^n-1} \mathbb{P}(|B_{j2^{-n}t} - B_{j2^{-n}}| \leqslant c2^{-n/2}\sqrt{n}) \\
&= |\mathbb{P}(|B_1| \leqslant c\sqrt{n})|^{2^n} \\
&\leqslant |1 - (\sqrt{2/\pi}e^{-(c\sqrt{n}+1)^2/2}|^{2^n} \\
&\leqslant \exp(-2^n(\sqrt{2/\pi}e^{-(c\sqrt{n}+1)^2/2}))\,.
\end{aligned}$$

Let us set $C_n = \sqrt{2\log 2} - 2n^{-1/2}$. Then we can form the events $A_n := \{M_n \leqslant C_n 2^{-n/2}\sqrt{n}\}$. By applying the previous estimate, we obtain that

$$\sum_{n=1}^{\infty} \mathbb{P}(M_n \leqslant C_n 2^{-n/2}\sqrt{n}) \leqslant \infty\,,$$

and hence by applying Borel-Cantelli we obtain that a.s.

$$\lim\inf_{n\to\infty} \frac{M_n}{\sqrt{2^{-n}n}} \geqslant \sqrt{2\log 2}\,.$$

Reversibly, we use the reflection principle for one-dimensional Brownian Motion, i.e.

$$\mathbb{P}(\sup_{0\leqslant s\leqslant t} B_s \geqslant a) = 2\mathbb{P}(B_t \geqslant a)\,.$$

For $c > \sqrt{2\log 2}$, we have that

$$\begin{aligned}\mathbb{P}(M_n \geqslant c2^{-\frac{n}{2}}\sqrt{n}) &\leqslant \mathbb{P}(\sup_{0\leqslant t\leqslant 2^{-n}} |B_t| \geqslant c2^{-n/2}\sqrt{n}) \\ &\leqslant 2^n 4\mathbb{P}(B_1 \geqslant c\sqrt{n}) \\ &\leqslant (4\sqrt{2\pi})2^n e^{-c^2 n/2} \\ &\leqslant (4\sqrt{2\pi})\exp(-n(c-\sqrt{2\log 2})^2/2)\,.\end{aligned}$$

If we choose $C_n = \sqrt{2\log 2} + n^{-1/4}$, we get that

$$\sum_{n=1}^{\infty} \mathbb{P}(M_n \geqslant C_n 2^{-n/2}\sqrt{n}) \leqslant \infty\,,$$

and hence by Borel-Cantelli, we have with probability 1 that

$$\limsup_{n\to\infty} \frac{M_n}{\sqrt{2^{-n}n}} \leqslant \sqrt{2\log 2}\,.$$

□

Estimate for the mapping-out function and the Distortion Theorem

Proposition A.2.1. *Let $S \subset \delta H$ be a measurable set on the Martin boundary of H. Then*

$$\lim_{y\to\infty, x/y\to 0} \pi y \mathbb{P}_{x+iy}(\hat{B}_{T(H)} \in S) = Leb(g_K(S))\,.$$

Proof. Consider the map $g_K(x+iy) = u+iv$. Then as $y \to \infty$ and $x/y \to 0$, we have that $u/y \to 0$ and $v/y \to 1$. Using the conformal invariance for of the Brownian motion and the (known) formula for the density of the harmonic measure in $\mathbb{H}$, we have that

$$\mathbb{P}_{x+iy}(\hat{B}_{T(H)} \in S) = \mathbb{P}_{u+iv}(B_{T(\mathbb{H})} \in g_K(S)) = \int_{g_K(S)} \frac{v}{\pi((t-u)^2+v^2)} dt\,.$$

□

A second result that we use is the Distortion Theorem for univalent functions. For this, we consider $\mathcal{S}$ to be the set of univalent functions on the unit disk that fix the origin and have $f'(0) = 1$. Note that Riemann Mapping Theorem tells us that we can study the simply connected domains by considering the study of univalent functions on $\mathbb{D}$. In the following, we present the following Distrotion Theorem that is used throught the Thesis. Its proof relies on an estimate that the reader can find in [**?**], Proposition 3.20.

Theorem A.2.2. *For $f \in \mathcal{S}$ and $z \in \mathbb{D}$, we have that*

$$\frac{1-|z|}{(1+|z|)^3} \leqslant |f'(z)| \leqslant \frac{1+|z|}{(1-|z|)^3}\,.$$

Proof. Due to the fact that $\mathcal{S}$ is closed under conjugation by rotations, it is enough to prove the estimate for $x \in (0,1)$ on the real line. Using Koebe 1/4 Theorem, we prove that for $f \in \mathbb{S}$, we have $|f'| \neq 0$ (for the precise statament , look at Corollary 3.19 in [?]. Since the derivative of a holomorphic function is holomorphic and $f'(0) = 1$, we can define an analytic function $\log f'$. i.e. we find h analytic such that $e^h = f'$. Using that $\log z = \log |z| + i \arg(z)$, we have that

$$x\partial_x \mathrm{Re}(h(x)) = \mathrm{Re}\left(\frac{xf"(x)}{f'(x)}\right).$$

Thus by the technical estimate in Proposition 3.20 we obtain that

$$\frac{2x-4}{1-x^2}\partial_x \log |f'(x)| \leqslant \frac{2x+4}{1-x^2},$$

and since $\log |f'(0)| = 0$ we can integrate the inequalities and after exponantiating we obtain the bounds.

□