Pathwise and probabilistic analysis in the context of Schramm-Loewner Evolutions (SLE)

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Outline

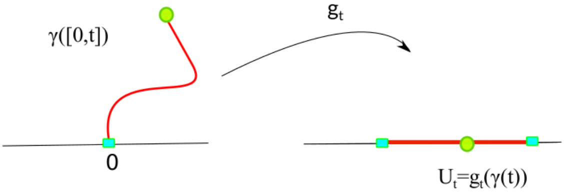
1. Introduction to SLE

2. Backward Bessel SDE and SLE
   - Are the conformal welding homeomorphisms continuous in κ?
   - New structural information about the backward SLE traces.

3. A radius independent SDE in the context of backward Loewner differential equation
   - The explicit law of the contangent of the argument of points under the backward Loewner flow.

Conformal maps and the Loewner equation

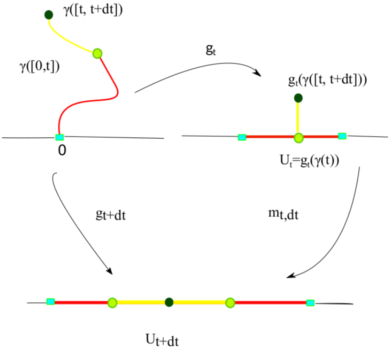
- In general, for a non-self crossing curve $\gamma(t) : [0, \infty) \to \bar{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(\infty) = \infty$, we consider the simply connected domain $\mathbb{H} \setminus \gamma([0, t])$.



- Using the Riemann Mapping Theorem for the simply connected domain $\mathbb{H} \setminus \gamma([0, t])$, we have a three parameter family of conformal maps $g_t : \mathbb{H} \setminus \gamma([0, t]) \to \mathbb{H}$.
- Loewner Equation encodes the dependence between the evolution of the maps g_t when the curve $\gamma([0, t])$ grows.

Conformal maps and the Loewner equation

- Is there a way to use g_t to find g_{t+dt} ?



- The precise dynamics is encoded by the Loewner Differential Equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z\,.$$

Definition of SLE

- To output random continous curves, U_t has to be a random continous driver. Moreover, the random driver U_t induces a law on the curves $\gamma(t) := \lim_{y\to 0+} g_t^{-1}(iy)$.

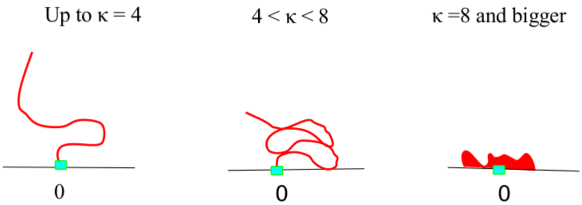
Definition (Schramm)

Let B_t be a standard real Brownian motion starting from 0. The chordal SLE(κ) is defined as the law on curves induced by the solution to the following ordinary differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z.$$

SLE phase transitions

- It is proved that there are two phase transitions when κ varies between 0 and ∞ .



- In order to show this, set $X_t(x) := \frac{g_t(x\sqrt{\kappa}) - \sqrt{\kappa}B_t}{\sqrt{\kappa}}$. Thus, for all $x \in \mathbb{R} \setminus \{0\}$ and all times t up to $X_t(x) \neq 0$, we have

$$X_t(x) = x + B_t + \int_0^t \frac{2/\kappa}{X_s} ds \, .$$

The Loewner Differential Equation

- The forward and backward Loewner Differential Equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \qquad \dot{h}_t(z) = \frac{-2}{h_t(z) - U_t}. \tag{1}$$

- For $U_t = \sqrt{\kappa} B_t$, via $h_t(z) - \sqrt{\kappa} B_t = z_t$, we obtain

$$dz_t = \frac{-2}{z_t} dt - \sqrt{\kappa} dB_t. \tag{2}$$

The backward Bessel SDE in the context of SLE

- Extending the maps to the real line, we obtain a backward Bessel process

$$dZ_t = \frac{-2/\kappa}{Z_t} dt + dB_t \,, \quad Z_0 = x \in \mathbb{R}.$$

- We have $d(\kappa) = 1 - 4/\kappa$. Thus,

$$d(\kappa) \leq 0, \qquad \kappa \leq 4, \text{ origin is absorbing,}$$

$$d(\kappa) > 0, \qquad \kappa > 4, \textbf{ origin is reflective}.$$

1) Are the conformal welding homeomorphisms continuous in κ?

2) Can we obtain some new structural information about the backward SLE_κ traces ?

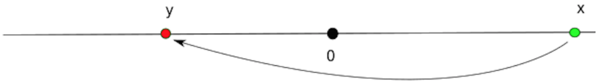
Continuity of the welding homeomorphism

- Previous work: S. Sheffield [2010]; K. Astala, P. Jones, A. Kupiainen, E. Saksman [2011]; S. Rohde and D. Zhan [2013]; W.Qian, J. Miller, Oliver McEnteggart [2018].

We study the conformal welding homeomorphism: two points $x > 0$ and $y < 0$ are to be identified if they hit zero simultaneously under the backward Loewner differential equation.

Theorem

The welding homeomorphism induced by the backward Loewner differential equation on the real line when driven by $\sqrt{\kappa}B_t$ is sequentially continuous in the parameter κ, for $\kappa \in [0, 8/3)$, a.s. everywhere except at most countably many points on the real axis.



Phase transition and structural information about the backward SLE traces

Theorem

- *For $\kappa \in [0,4]$, for any $t \in [0,T]$, a.s. there is a unique solution of the backward Loewner differential equation started from the origin.*
- *For $\kappa > 4$ a.s. there are at least two solutions.*
- *For $\kappa \in (4,\infty)$, on macroscopic excursions from the origin of the squared Bessel process obtained from extensions of the backward SLE_κ maps on the real line, we obtain macroscopic hulls and macroscopic double points of the backward SLE_κ trace.*

- Key point: Low dimensional information - excursion theory of Bessel processes give structural information about the traces.
- Corollary: a.s. simpleness/non-simpleness of the trace equivalent with behaviour of Bessel at origin (related with question posed by Prof. Peter Friz).

A time change of the Loewner differential equation

- For $z_t = x_t + iy_t$ we obtain the coupled equations

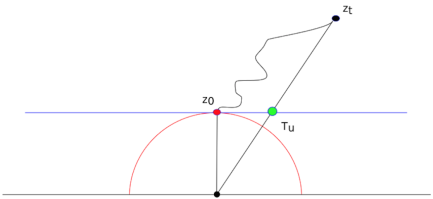
$$dx_t = \frac{-2x_t}{x_t^2 + y_t^2} dt - \sqrt{\kappa} dB_t \,, \qquad dy_t = \frac{2y_t}{x_t^2 + y_t^2} dt \,.$$

Furthermore,

$$d\frac{x_t}{y_t} = -\frac{\frac{4x_t}{y_t}}{x_t^2 + y_t^2} dt - \frac{\sqrt{\kappa} dB_t}{y_t} \,.$$

- Using $u(s) = \int_0^s \frac{dt}{y_t^2}$, $\tilde{B}_{u(s)} = \int_0^s \frac{dB_t}{y_t}$, we obtain the following SDE

$$dT_u = -4 \frac{T_u}{1 + \kappa T_u^2} du + d\tilde{B}_u \,. \tag{3}$$



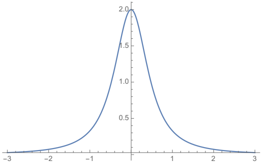
A new framework and the result

Theorem

Let us consider the functions $f \in L^1(\mu)$, *where* $\mu(dx) = \frac{dx}{(1+x^2)^{4/\kappa}}$.
The process T_u *has stationary distribution with density* $\rho(T) = C\frac{1}{(\kappa T^2+1)^{4/\kappa}}$, $\rho(T) \sim \frac{1}{T^{8/\kappa}}$ *as* $T \to \infty$. *For* $\kappa < 8$, *a.s.*

$$\left|\frac{1}{u(S,\omega)}\int_0^{u(S,\omega)} f(T_s(\omega))ds - \mu(f)\right| \xrightarrow{S\to\infty} 0.$$

For $\kappa < 8$, *we obtain family of random times* $s(\omega)$ *on which* $ctg(\arg(h_{s(\omega)}(i))) \to \mu(f)$.



Current and future projects

- Quasi-sure Stochastic Analysis and the stability properties of SLE traces and related objects.

$$|\tilde{h}_t'(z_0)| = e^{-at} \exp\left(2a \int_0^t \frac{\tilde{K}_s^2 + 1}{\tilde{K}_s^2 - 1} ds\right), \; d\tilde{K}_t = 2a\tilde{K}_t dt + \sqrt{1 + \tilde{K}_t^2} d\tilde{B}_t.$$

  [Also interesting in the context of Growth Models.]

- New framework in Rough Path Theory: Pathwise analysis of Singular RDEs (LDE). [soon visit MPI]

- Random Matrix Theory and multiple radial SLEs :

- George A., Jianping J., Joseba D., Eric E.;

Thank you for your attention!